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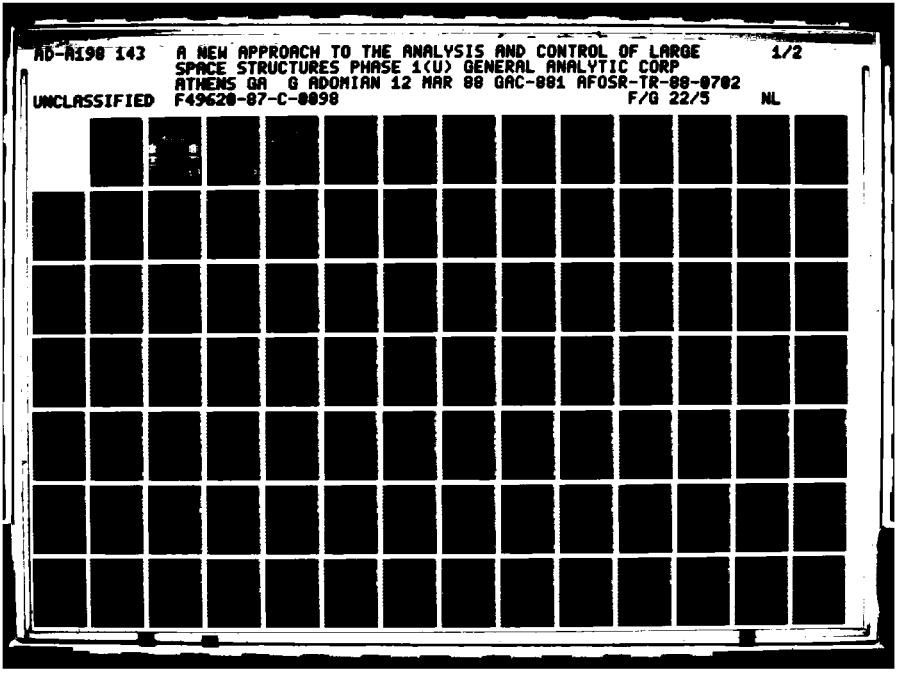

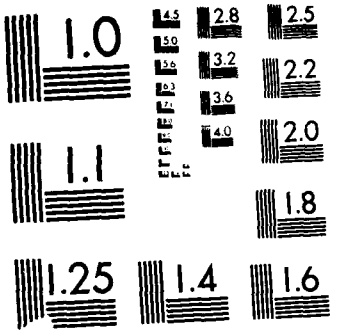

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A NEW APPROACH TO THE ANALYSIS AND CONTROL OF LARGE SPACE STRUCTURES

S.B.I.R. Phase I

Air Force Contract F49 620-87-C-0098
October 15, 1987 - April 15, 1988

Dr. Anthony K. Amos, Project Manager

Technical Report G.A.C. 881

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Nonlinear ordinary and partial differential equations modelling problems of vibration, heating, and control systems are analyzed using the author's decomposition method. The new approach is also applied to nonlinear stochastic distributed control systems using the Hamiltonian formulation of control theory.

The matrix Riccati equation is also solved, providing another approach to control theory.

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OBJECTIVES AND ACCOMPLISHMENTS

OBJECTIVE OF PROJECT

Demonstrate use and applicability of decomposition method to typical model equations of problems relevant to space structures such as vibration and heating

ACCOMPLISHMENTS

1) *Fulfillment of project objective*
2) *Further insights and progress*
   a) *Method is global applying to parabolic, elliptic, and hyperbolic equations and therefore widely applicable to aerospace problems*
   b) *Method can provide a new approach to control theory for nonlinear stochastic multidimensional cases in space structure applications*

ADVANTAGES OF THE DECOMPOSITION METHOD

1) *Drastically reduced computation time*
2) *No restrictive assumptions changing physical problem*
   a) *No linearization of nonlinear problems*
   b) *No limitation to perturbation theory*
   c) *No closure approximation*
3) *Easier to see functional relationships*
4) *Global procedure (same method for any class of equations)*

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EXECUTIVE SUMMARY - PHASE I

In the design and operation of large space structures, the effects of time dependent and random vibrations, heating and cooling, control forces, etc., lead to severe analytical problems because of nonlinearities, time-varying effects such as self-shadowing in trusses, couplings, etc. These problems lead to model equations which can be differential, partial differential, or integro-differential equations involving complex nonlinearities and stochastic parameters.

Our objective is to identify these problems and to demonstrate that the Adomian decomposition method provides a new useful approach to such problems and that the approach provides physically more realistic and correct solutions, and, further, that tremendous computational effort can be avoided.

The saving of computer time results from the avoidance of discretized methods and the ability to obtain a continuous analytical solution. This not only saves computation time but makes it easier to see functional dependences.

The fact that solutions are necessarily more correct arises from avoidance of the conventional linearization and restrictive assumptions normally required. The ease and accuracy of the method is demonstrated in a variety of selected examples and problems.

In addition to the fulfillment of the above contract objective, some very significant other and additional new insights and progress have been achieved to be expanded on in Phase II. We have established that

parabolic, elliptic, and hyperbolic equations can all be solved by the decomposition method (there is no other method that can do this analytically.

Secondly we have seen as a direct result of this study that we can generalize Kalman filters to a continuous nonlinear stochastic case without white noise limitations and that we can generalize control theory analytically to nonlinear stochastic multidimensional control theory for space structures, an insight of the greatest significance to be an important area of our Phase II proposed work.

PUBLICATIONS IN REVIEWED JOURNALS
CREDITED TO CONTRACT

1) In the *Journal of American Institute of Aeronautics and Astronautics*

"A New Approach to the Analyses and Control of Large Space Structures," G. Adomian, to appear in 1988.

2) In the *Journal of American Institute of Aeronautics and Astronautics*

"Model Equations of Gas Dynamics," G. Adomian, to appear in 1988.

3) In the *Journal of Mathematical Analyses and Applications*

"A Review of the Decomposition Method," G. Adomian, to appear Sept. 1988.

Principal accomplishments:

1) The contract statement of work has been fulfilled in establishing that the decomposition method is valuable in the basic types of typical model equations that will arise in analysis of large space structures such as vibration, heating, etc. It has been shown that the decomposition method solves typical nonlinear and stochastic and multidimensional model equations without customary restrictive and limiting assumptions or procedures involving excessive computation.

2) Additional insights as a result of this study are:

   a) the procedure will apply to parabolic, elliptic, and hyperbolic equations, a unique feature and advantage of the decomposition method, which means rather global applicability to aerospace problems.

   b) Control theory can be generalized to nonlinear stochastic and multidimensional cases for application to space structures and will be studied in detail in Phase II. Kalman filter theory can similarly be generalized to the case of nonlinear stochastic operators in both state and measurement equations.

INTRODUCTION

The large structures contemplated would be constructed in space. Because of the limitations on launching massive payloads, it is clear that these structures will be made of lightweight material and will necessarily be flexible and easily excited into vibrations.

Analytical problems will arise in designing large space structures in which physically realistic and accurate solutions will be critical. Such designs must consider weight, sizes, stiffness, thermal and mechanical distortions, stresses due to gravity and positioning thrusts. Some specific analytical problems will involve vibration, heating and cooling, multidimensional control, and structural problems arising from random support motion and random fluctuations of the system dynamic properties. → (14 > 3) ⇒

In large space structures, vibrations can result from machinery, thermal transients, and operational maneuvers. The response to such vibrations depends on structural parameters (stiffness, mass, damping) and the ex ernal excitations and on the control system.

Vibration is hardly the only problem. There will be effects of severe heating (e.g., on sun-illuminated parts) and severe cooling on dark sides with resulting exapnsions, contractions, and stresses on the structure. Additionally there will be the problem of nonlinear stochastic multidimensional control of a large complex structure with limited rigidity, and the interactions through the central system as a result of frequencies introduced in the control loop through rotation, bending,

expansion and contraction. The means employed for passive damping also complicate matters since modifications of mass, stiffness, and damping affect the control design.

If a flexible structure were designed to be rotated at a constant angular velocity, then complex Coriolis and centripetal accelerations as well as accelerations due to structural deformations would have to be taken into account. Thus the rotational motions are a factor in dynamic analysis. Ordinarily, such analyses immediately rely on linearizing the governing equations of motion. The decomposition method, on the other hand, allows us to avoid this restrictive procedure.

The fact that the structural materials will be nonhomogeneous composites will result in random fluctuations in dynamic properties. Furthermore, once the structure begins to vibrate, structural parameters may fluctuate. Thus one must be able to deal with cases in which the stiffness or damping are random functions of time rather than constants. The equations involved will then be stochastic differential equations in the sense used by the principal investigator in the referenced works*, i.e., differential equations with stochastic process coefficients rather than the much simpler case with only stochastic inputs - also called s.d.e. by many investigators.

In the so-called "disordered systems" where parameters are random variables** we have random boundary-value problems. The decomposition method can deal with boundary operator equations which are random, nonlinear, or even coupled.

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*See particularly references [1,2].

**Assuming that systems parameters are merely uncertain rather than fluctuating in time.

One should consider various types of nonlinearities in order to gain insights into possible unusual or growing structural vibrations that might lead to failure. Thus our ability, when using decomposition, to consider strong and composite nonlinearities is of extreme importance.

Introduction to the Decomposition Method:

Real physical systems are nonlinear and stochastic; linearity and determinism are special cases. Realistic modeling will lead to systems of ordinary or partial (nonlinear stochastic) differential equations. Our model equations will involve nonlinearities and/or stochastic parameters, inputs, and initial/boundary conditions.

The methods that have been available so far to solve such systems have limitations which require close examination. It has become standard to linearize nonlinear equations or to assume they are "weakly nonlinear." In the stochastic case, resort to statistical linearization is a common procedure. Stochasticity in parameters is quite commonly ignored. Inputs which we know to be stochastic processes are assumed to be delta-correlated or to represent "small" fluctuations. Thus unless systems are essentially linear and deterministic, i.e., unless nonlinearity and stochasticity are not significant factors, what is commonly being done - whether apparent or not - is to change the system physical model to a mathematically more tractable one within the capability of the available mathematics. The contrast between "closed-form solutions" and "approximate solutions" has been exaggerated and, sometimes, not even clearly understood. The reason is simple. All modelling is approximation. In modelling, we always have a compromise between realistic modeling and

tractability in our available methodologies. The model actually solved by conventional methods is often not the model we had wanted to solve - which was, of course, already and necessarily an approximation to reality. With the model forced into a tractable form by restrictive assumptions and techniques such as linearization, one can then find "nice" closed-form solutions. However this only means we have an identifiable or well-known series such as $\sin x = x - x^3/3! + \cdots$ as the solution for a mathematically simplified problem.

Our method of attack is a continuous approximation method which yields a rapidly converging series solution* (which can sometimes be summed into a familiar closed form).

What is significant is that it does not require linearization. It solves nonlinear problems. We do not limit stochastic cases to perturbation theory, Wiener processes, delta-correlated processes, Markovian assumptions, closure approximations, or quasi-monochromatic assumptions.

This method is computationally convenient. It does not require discretization into grids with its resulting massive computation and approximation between computing points. It yields a continuous analytic solution into which we then put numbers. We can then see the rapid stabilization of the solution as terms are calculated.

Such a methodology - we have called it the "decomposition method" - which yields solution for a model retaining actual nonlinearity and stochasticity, seems preferable to a nice mathematical solution - a so-called exact solution - to an artificial or simplified problem in which one

*A proof of the convergence has now been made by the principal investigator and is in publication.

linearizes or assumes unphysical processes. For example, most nonlinear stochastic oscillator problems assume only stochastic excitation and require hierarchy methods and closure approximations. However, such assumptions have been shown to be equivalent to perturbation methods.** In a case where the perturbation method is not adequate to the problem, e.g., where fluctuations are large, decomposition offers a more accurate solution. Decomposition is the more general method always resulting in a correct solution.** In the cases where perturbative methods have been adequate, or linear and/or deterministic solutions would be sufficient, decomposition will yield the *same* solution.

In large (flexible) space structures a central problem will be random, as well as periodic, vibrations. For random vibrations, often the usual treatments, such as the Fokker-Planck-Kolmogorov, perturbation, statistical linearization, Gram-Charlier expansions, averaging and cumulant closure methods, etc., are less than desirable for many reasons (particularly more limitations and less accuracy in many real problems). Restrictive approximations that are made, e.g., white noise excitation and deterministic parameters, Wiener processes, Gaussian behavior, etc., may well result in mathematical solutions which are incomplete or which deviate from the actual physical behavior.

We believe the decomposition method offers a new approach with a significant potential for major contributions not possible otherwise because of the avoidance of restrictions necessary in other approaches.

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*See reference [1].

**See references [1] and [2] .

ILLUSTRATIONS OF THE DECOMPOSITION METHOD

Solution of nonlinear equations arising in the modeling of a physical system generally begins with some form of linearization, assumptions of weak nonlinearity, and "smallness." Yet physical systems are nonlinear, and real systems generally involve random fluctuations. The general case is a nonlinear stochastic system, and "linear" and "deterministic" are *special* cases. We aim to solve physical problems realistically, not simply artificial versions of those problems selected so as to make the mathematics tractable. It is true that we have gone a long way with the earlier methods, and in many problems they are completely adequate; in others, they are a good first approximation to gain some insight. Yet, one can find cases in which they are not adequate; clearly not all systems are linear and deterministic. Further, all effects and behaviors in complex systems are not instantaneous, and a serious attempt to explain behavior of a complex dynamical system, and eventually to control it, must sometimes consider delays or retarded effects.

To solve such problems, we must be able to solve equations or systems of equations which may involve differential or partial differential operators, linear or nonlinear, deterministic or stochastic, and possibly involving delays.

As ambitious as it may appear, it is fortuitous that the decomposition method appears to be capable of such solutions in a fairly wide class of problems. The superficial resemblance of this method to some other methods can be misleading; the decomposition method solves problems not solvable by other methods, or which are only solvable with much more

difficulty or computation. The method is an "approximation" method, not a "closed form" solution. The usual significance of these terms is that in the one case, we have an exact answer and in the other an approximate one. Clearly, however, a method of solution which changes the problem to a *different, easier mathematical problem* and then solves it exactly is not to be preferred to one in which the *actual* nonlinear and/or stochastic model is treated with an "approximate" method which provides accurate, rapidly convergent, and computable series of terms.

Let us consider how the decomposition method may change the situation. Consider first a deterministic ordinary differential equation $Fu = x(t)$ where Fu is given as the sum of a linear term and a nonlinear term. The linear term is $Lu + Ru$, where L is the highest-ordered derivative and R is the remainder of the linear operator. If Nu is the nonlinear term, $Lu + Ru + Nu = x(t)$, which we write $Lu = x - Ru - Nu$. If this is an initial-condition problem, then for $L = d^n/dt^n$, we define L^{-1} as the n-fold definite integration from 0 to t.

From $L^{-1}Lu = L^{-1}x - L^{-1}Ru - L^{-1}Nu$ we obtain

$$u = u_0 - L^{-1}Ru - L^{-1}Nu,$$

where $u_0 = L^{-1}x + u(0) + tu'(0) + \cdots + t^{n-1}u^{(n-1)}(0)/(n-1)!$ Now u is decomposed into $\sum_{n=0}^{\infty} u_n$ and Nu is written $\sum_{n=0}^{\infty} A_n(u_0,u_1,\ldots,u_n)$ where the A_n are the Adomian polynomials generated fro the specific nonlinearity to represent it exactly [23]. These polynomials have now been discussed in numerous papers and books and are listed for convenience

at the end of this section*. Now, if $Nu = f(u)$ is an analytic function, we can write $f(u) = \sum_{n=0}^{\infty} A_n$ in a series which generally converges quite rapidly**. Now $u = u_0 - L^{-1}R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n$ and we determine $u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n$ for $n \geq 0$ to find all components of u, provided we have the appropriate integrability. The A_n depend on the specific nonlinearity. Further, A_0 depends only on u_0; A_1 depends on u_0, u_1, etc. This makes the solution calculable.

If boundary conditions are specified instead of initial conditions, we would allow L^{-1} to be n-fold indefinite integrations, then evaluate the constants.

The existence, uniqueness, and other properties of the solution depend on the terms in u_0. In the linear case where Nu vanishes we have $u = u_0 - L^{-1}Ru_0 - L^{-1}Ru_1 - \cdots = u_0 - L^{-1}Ru_0 + (L^{-1}R)(L^{-1}R)u_0 - \cdots$ or $u = \sum_{n=0}^{\infty} (-1)^n (L^{-1}R)^n u_0$. If the given conditions are zero, $u_0 = L^{-1}g$ and $u = \sum_{n=0}^{\infty} (-1)^n (L^{-1}R)^n L^{-1}g$. Thus $Fu = g$ becomes $u = F^{-1}g$ where the inverse is $F^{-1} = \sum_{n=0}^{\infty} (-1)^n (L^{-1}R)^n L^{-1}$. This is, of course, a succinct introduction; general cases are discussed elsewhere [1-3].

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*We have listed A_n for $n = 0,1,2,\ldots,10$. References [1,2,3,23] show derivations.

**The $\sum A_n$ is a rearranged generalized Taylor series. See Nonlinear Stochastic Systems Theory and Applications to Physics, G. Adomian, Reidel publishing Co., 1988.

Example: As a simple (linear deterministic) example, consider the well-known Airy's Equation $y'' - ty = 0$ which arise in applications in mathematical physics and are useful for asymptotic representations of various special functions. Assume conditions $y(0) = 1$ and $y'(0) = 1$ and write the equation in the form $Ly - Ry = 0$ with $L = d^2/dt^2$, $R = t$, $L^{-1} = \int_0^t (\int_0^t [\cdot]dt)dt$. Then operating with L^{-1} we obtain $y(t) = y(0) + ty'(0) + L^{-1}Ry$. Thus

$$y_0 = 1 + t$$

$$y_1 = L^{-1}Ry_0 = L^{-1}t(1 + t) = \frac{t^3}{2\cdot 3} + \frac{t^4}{3\cdot 4} = \frac{1\cdot t^3}{3!} + \frac{2\cdot t^4}{4!} \qquad (1)$$

$$y_2 = L^{-1}Ry_1 = \frac{t^6}{2\cdot 3\cdot 5\cdot 6} + \frac{t^7}{3\cdot 4\cdot 6\cdot 7} = \frac{1\cdot 4\cdot 6\cdot t^6}{6!} + \frac{2\cdot 5\cdot t^7}{7!}$$

$$\vdots$$

$$y_n = \frac{1\cdot 4\cdot 7\ldots(3n-2)t^{3n}}{(3n)!} + \frac{2\cdot 5\cdot 8\ldots(3n-1)t^{3n+1}}{(3n+1)!}$$

The sum $\sum_{n=0}^{\infty} y_n$ is the solution y. For $t = 0.1$, one term is sufficient for six decimal accuracy. For $t = 1.0$, three terms are sufficient to at least three digits.

Example: Consider the equation $d^2u/dx^2 - kx^pu = g$ with $u(1) = u(-1) = 0$. Using the decomposition method [1], let $L = d^2/dx^2$ and $Lu = g + kx^pu$. Operating with L^{-1}, we have $L^{-1}Lu = L^{-1}g + L^{-1}kx^pu$. Then

$$u = c_1 + c_2x + gx^2/2 + L^{-1}kx^pu \qquad (2)$$

Let $u = \sum_{n=0}^{\infty} u_n$ with $u_0 = c_1 + c_2x + gx^2/2$. Then $u_{m+1} = L^{-1}kx^pu_m$ with $m \geq 0$. Thus $u = \sum_{m=0}^{\infty} (L^{-1}kx^p)^m u_0$ or $u = \sum_{m=0}^{\infty} (L^{-1}kx^p)^m c_1 + \sum_{m=0}^{\infty} (L^{-1}kx^p)^m c_2x +$

$\sum_{m=0}^{\infty} (L^{-1}kx^p)^m gx^2/2$, and finally $u = c_1\Phi_1(x) + c_2\Phi_2(x) + \Gamma(x)$ where

$$\Phi_1(x) = \sum_{m=0}^{\infty} k^m x^{mp+2m}/(mp + 2m - 1)(mp + 2m)$$

$$\Phi_2(x) = \sum_{m=0}^{\infty} k^m x^{mp+2m+1}/(mp + 2m)(mp + 2m - 1) \qquad (3)$$

$$\Gamma(x) = \sum_{m=0}^{\infty} (1/2)gk^m x^{mp+2m+2}/(mp + 2m + 1)(mp + 2m + 2)$$

Since $u(1) = u(-1) = 0$ we have $c_1\Phi_1(1) + c_2\Phi_2(1) + \Gamma(1) = 0$ and $c_1\Phi_1(-1) + c_2\Phi_2(-1) + \Gamma(-1) = 0$ or

$$\begin{bmatrix} \phi_1(1) & \phi_2(1) \\ \phi_1(-1) & \phi_2(-1) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\Gamma(1) \\ -\Gamma(-1) \end{bmatrix} \qquad (4)$$

or $\phi C = \Gamma$ or $C = \phi^{-1}\Gamma$ with

$$\phi^{-1} = \frac{\begin{bmatrix} \phi_2(-1) & -\phi_2(1) \\ -\phi_1(-1) & \phi_1(1) \end{bmatrix}}{\phi_1(1)\phi_2(-1) - \phi_2(1)\phi_1(-1)}$$

Thus

$$c_1 = [\phi_2(1)\Gamma(-1) - \phi_2(-1)\Gamma(1)]/[\phi_1(1)\phi_2(-1) - \phi_2(1)\phi_1(-1)]$$

$$c_2 = [\phi_1(-1)\Gamma(1) - \phi_1(1)\Gamma(-1)]/[\phi_1(1)\phi_2(-1) - \phi_2(1)\phi_1(-1)]$$

and the complete solution can now be determined.

Suppose in the above example, we let $k = 40$, $p = 1$, $g = 2$. Thus we consider the equation

$$d^2u/dx^2 - 40xu = 2$$

with $u(-1) = u(1) = 0$. This is the one-dimensional case of the elliptic equation $\Delta^2 u = f(x,y,z) + k(x,y,z)u$ arising in problems of physics and engineering.

Here let $L = d^2/dx^2$ and we have $Lx = 2 + 40xu$.

This is a relatively stiff case because of the large coefficient of u, and the nonzero forcing function yields an additional Airy-like function. Operating with L^{-1} yields

$$u = A + Bx + L^{-1}(2) + L^{-1}(40xu).$$

Let

$$u_0 = A + Bx + L^{-1}(2) = A + Bx + x^2$$

and let $u = \sum_{n=0}^{\infty} u_n$ with the components to be determined so that the sum is u.

We identify $u_{n+1} = L^{-1}(40xu_n)$. Then all components can be determined, e.g.,

$$u_1 = (20/3)Ax^3 + (10/3)Bx^4 + 2x^5$$

and

$$u_2 = (80/9)Ax^6 + (200/63)Bx^7 + (10/7)x^8,$$

etc. An n-term approximant $\phi_n = \sum_{i=0}^{n-1} u_i$ with $n = 12$ for values of x is given in the table below.

n-term approximant ϕ_n for $n = 12$

x	ϕ_{12}
0.2	-0.135649
0.4	-0.113969
0.6	-0.083321
0.8	-0.050944
1.0	0

These easily-obtained results for a 12 term approximation are exact to seven digit accuracy.

The solution is found just as easily for nonlinear versions without linearization by simply using the A_n polynomials for the nonlinear terms.

APPROXIMATION OF NONLINEARITIES USING A_n POLYNOMIALS

Using the A_n polynomials, there is no need for mathematically inadequate and physically unrealistic approximations or linearizations. Thus, if the modeling retains the inherent nonlinearities, we may expect solutions conforming much more closely to actual behavior.

Consider simple nonlinear operators not involving differentials, i.e., of the form $Ny = f(y)$.

For polynomial nonlinearities the A_n being sums of various products of the y_i up to $i = n$ can be written in symmetrized form. Thus, if we have $Ny = y^2 = \sum_{n=0}^{\infty} A_n$, $A_0 = y_0^2$, $A_1 = 2y_0y_1$, $A_2 = y_1^2 + 2y_0y_2$, etc.; but we can write this as $A_0 = y_0y_0$, $A_1 = y_0y_1 + y_1y_0$, $A_2 = y_0y_2 + y_1y_1 + y_2y_0$, etc., i.e., the first subscript goes from 0 to n, and the second is chosen such that the sum of subscripts is n.

In extending to any analytic function $f(y)$ we define $h_n(y) = d^nf/dy^n$ and write a convenient heuristic rule which users may find more convenient than mathematical derivations -

$$A_n = \sum_{\nu=1}^{n} c(\nu,n)h_\nu(y_0)$$

For example,

$$A_3 = c(1,3)h_1 + c(2,3)h_2 + c(3,3)h_3$$

The $c(\nu,n)$ can be obtained by simply asking how ma-y combinations of ν integers add to n. Thus $c(\nu,n)$ will mean the sum (from 1 to ν) of the products of ν of the y_i terms whose subscripts add to n. To get $c(2,3)$, we see two integers can add to 3 only if one integer is 1 and the other is 2 (if zero is excluded). Hence, we write $c(2,3) = y_1y_2$. To get $c(1,3)$,

the coefficient of $h_1(y_0)$, we have one y_i and its subscript must be 3, hence $c(1,3) = y_3$. What about $c(3,3)$, the coefficient of $h_3(y_0)$? Now we need 3 factors y_i with subscripts summing to 3, hence each subscript must be 1 and $c(3,3) = y_1y_1y_1 = y_1^3$. This is not quite right, and we add another heuristic rule. If we have repetitions of subscripts we divide by the factorial of the number of repetitions. Then, $c(3,3) = (1/3!)y_1^3$. We now have

$$A_3 = h_1(y_0)y_3 + h_2(y_0)y_1y_2 + h_3(y_0)(1/3!)y_1^3$$

To write A_6, for example, we need the coefficients for the terms $h_\nu(y_0)$ for ν from 1 to 6. The coefficient of h_6 must involve six integers adding to 6 or y_1^6 hence the coefficient of $h_6(y_0)$ is $(1/6!)y_1^6$. What about the coefficient for $h_2(y_0)$ in A_6 or $v = 2$, $n = 6$? Clearly we need two integers that sum to 6. These are (1,5), (2,4), and (3,3). Thus, the coefficient $c(2,6)$ is $(1/2!)y_3^2 + y_2y_4 + y_1y_5$. The terms involve $\Pi_{i=1}^{\nu} y_{k_i}$ with $\sum_{i=1}^{n} k_i = n$, and if we have j repeated subscripts, we divide by $j!$:

$$A_0 = h_0(y_0)$$

$$A_1 = h_1(y_0)y_1$$

$$A_2 = h_1(y_0)y_2 + h_2(y_0)(1/2!)y_1^2$$

$$A_3 = h_1(y_0)y_3 + h_2(y_0)y_1y_2 + h_3(y_0)(1/3!)y_1^3$$

$$A_4 = h_1(y_0)y_4 + h_2(y_0)[(1/2!)y_2^2 + y_1y_3]$$
$$+ h_3(y_0)(1/2!)y_1^2y_2 + h_4(y_0)(1/4!)y_1^4$$

$$A_5 = h_1(y_0)y_5 + h_2(y_0)[y_2y_3 + y_1y_4]$$
$$+ h_3(y_0)[y_1(1/2!)y_2^2 + (1/2!)y_1^2y_3]$$
$$+ h_4(y_0)(1/3!)y_1^3y_2 + h_5(y_0)(1/5!)y_1^5$$

$$A_6 = h_1(y_0)y_6 + h_2(y_0)[(1/2!)y_3^2 + y_2y_4 + y_1y_5]$$

$$+ h_3(y_0)[(1/3!)y_2^3 + y_1y_2y_3 + (1/2!)y_1^2y_4]$$

$$+ h_4(y_0)[(1/2!)y_1^2(1/2!)y_2^2 + (1/3!)y_1^3y_3]$$

$$+ h_5(y_0)(1/4!)y_1^4y_2 + h_6(y_0)(1/6!)y_1^6$$

$$A_7 = h_1(y_0)y_7 + h_2(y_0)[y_3y_4 + y_2y_5 + y_1y_6]$$

$$+ h_3(y_0)[(1/2!)y_2^2y_3 + y_1(1/2!)y_3^2 + y_1y_2y_4 + (1/2!)y_1^2y_5]$$

$$+ h_4(y_0)[y_1(1/3!)y_2^3 + (1/2!)y_1^2y_2y_3 + (1/3!)y_1^3y_4]$$

$$+ h_5(y_0)[(1/3!)y_1^3(1/2!)y_2^2 + (1/4!)y_1^4y_3]$$

$$+ h_6(y_0)(1/5!)y_1^5y_2 + h_7(y_0)(1/7!)y_1^7$$

$$A_8 = h_1(y_0)y_8 + h_2(y_0)[(1/2!)y_4^2 + y_3y_5 + y_2y_6 + y_1y_7]$$

$$+ h_3(y_0)[y_2(1/2!)y_3^2 + (1/2!)y_2^2y_4 + y_1y_3y_4 + y_1y_2y_5 + (1/2!)y_1^2y_6]$$

$$+ h_4(y_0)[(1/4!)y_2^4 + y_1(1/2!)y_2^2y_3 + (1/2!)y_1^2(1/2!)y_3^2$$

$$+ (1/2!)y_1^2y_2y_4 + (1/3!)y_1^3y_5]$$

$$+ h_5(y_0)[(1/2!)y_1^2(1/3!)y_2^3 + (1/3!)y_1^3y_2y_3 + (1/4!)y_1^1y_4]$$

$$+ h_6(y_0)[(1/4!)y_1^4(1/2!)y_2^2 + (1/5!)y_1^5y_3]$$

$$+ h_7(y_0)(1/6!)y_1^6y_2 + h_8(y_0)(1/8!)y_1^8$$

$$A_9 = h_1(y_0)y_9 + h_2(y_0[y_4y_5 + y_3y_6 + y_2y_7 + y_1y_8]$$

$$+ h_3(y_0)[(1/3!)y_3^3 + y_2y_3y_4 + (1/2!)y_2^2y_5 + y_1(1/2!)y_4^2$$

$$+ y_1y_3y_5 + y_1y_2y_6 + (1/2!)y_1^2y_7]$$

$$+ h_4(y_0)[(1/3!)y_2^3y_3 + y_1y_2(1/2!)y_3^2 + y_1(1/2!)y_2^2y_4$$

$$+ (1/2!)y_1^2y_3y_4 + (1/2!)y_1^2y_2y_5 + (1/3!)y_1^3y_6]$$

$$+ h_5(y_0)[y_1(1/4!)y_2^4 + (1/2!)y_1^2(1/2!)y_2^2y_3$$

$$+ (1/3!)y_1^3(1/2!)y_3^2 + (1/3!)y_1^3y_2y_4 + (1/4!)y_1^4y_5]$$

$$+ h_6(y_0)[(1/3!)y_1^3(1/3!)y_2^3 + (1/4!)y_1^4y_2y_3 + (1/5!)y_1^5y_4]$$

$$+ h_7(y_0)[(1/5!)y_1^5(1/2!)y_2^2 + (1/6!)y_1^6y_3]$$

$$+ h_8(y_0)(1/7!)y_1^7y_2 + h_9(y_0)(1/9!)y_1^9$$

$$A_{10} = h_1(y_0)y_{10} + h_2(y_0)[(1/2!)y_5^2 + y_4y_6 + y_3y_7 + y_2y_8 + y_1y_9]$$

$$+ h_3(y_0)[(1/2!)y_3^2y_4 + y_2(1/2!)y_4^2 + y_2y_3y_5 + (1/2!)y_2^2y_6$$

$$+ h_1(y_0)y_4y_5 + y_1y_3y_6 + y_1y_2y_7 + (1/2!)y_1^2y_8]$$

$$+ h_4(y_0)[(1/2!)y_2^2(1/2!)y_3^2 + (1/3!)y_2^3y_4 + y_1(1/3!)y_3^3$$

$$+ y_1y_2y_3y_4 + y_1(1/2!)y_2^2y_5 + (1/2!)y_1^2(1/2!)y_4^2$$

$$+ (1/2!)y_1^2y_3y_5 + (1/2!)y_1^2y_2y_6 + (1/3!)y_1^3y_7]$$

$$+ h_5(y_0)[(1/5!)y_2^5 + y_1(1/3!)y_2^3y_3 + (1/2!)y_1^2y_2(1/2!)y_3^2$$

$$+ (1/2!)y_1^2(1/2!)y_2^2y_4 + (1/3!)y_1^3y_3y_4$$

$$+ (1/3!)y_1^3y_2y_5 + (1/4!)y_1^4y_6]$$

$$+ h_6(y_0)[(1/2!)y_1^2(1/4!)y_2^4 + (1/3!)y_1^3(1/2!)y_2^2y_3$$

$$+ (1/4!)y_1^4(1/2!)y_3^2 + (1/4!)y_1^4y_2y_4 + (1/5!)y_1^5y_5]$$

$$+ h_7(y_0)[(1/4!)y_1^4(1/3!)y_2^3 + (1/5!)y_1^5y_2y_3 + (1/6!)y_1^6y_4]$$

$$+ h_8(y_0)[(1/6!)y_1^6(1/2!)y_2^2 + (1/7!)y_1^7y_3]$$

$$+ h_9(y_0)(1/8!)y_1^8y_2 + h_{10}(y_0)(1/10!)y_1^{10}$$

Recent work has established that $\sum_{n=0}^{\infty} A_n$ forms a generalized Taylor series about the function $u_0(x)$.

Computational Time: In numerical solutions of physical problems, it is common to make computations at discrete space or time intervals. Computer methods are based on changing continuous problems to discrete problems. Thus, in solving a differential equation, one must solve the equation at each point of time.

Since these points must be close together to approximate the total solution, massive computations are needed and the resulting numerical printouts yield little insight into dependences. Another drawback of these methods is the fact that they involve *linearized approximations* in each interval. As the mesh is made finer to increase accuracy the number of computations can become enormous.

For example, solving even the simple linear equation

$$d^2y/dt^2 + dy/dt + y = 0$$

with $y(0) = 1$ and $y'(0) = 0$ requires a *thousand* solutions of the above equation or 8,000 actual computations to get the value of y every .01 seconds for 10 seconds. For one minute this means 6,000 solutions or 48,000 actual computations. Note that if one considers an equation in two independent variables x,t we require small steps in both variables. Then, the computations can go up by several orders of magnitude.

One can easily visualize a problem leading to a billion coupled difference equations to solve. For a simple scalar elliptic equation, we have one unknown at each mesh point. For more complex problems, there can be many unknowns at each mesh point and the resulting systems of difference equations (instead of being linear as in the previous case) may be nonlinear, time-dependent, and very large (inclusion of stochastic coefficients, etc., is still another matter). To solve massive systems, iterative procedures are used to solve simpler systems, then substitution to get "residuals" and repetitions of the process to produce corrections and until the error is (or is felt to be) within tolerable limits. To get accuracy the mesh must become very fine and computations required finally exceed any conceivable computer capability for complicated equations in x,y,z,t.

Solution by the decomposition method, on the other hand, is *continuous*, analytical, and requires no discretization; it corresponds to the results obtained by an infinite number of computations by discretization - *no linearization is involved, and the solution is accurate*. If variable coefficients, several independent variables, and nonlinearities are involved, the decomposition method is clearly preferable. In the case of stochastic equations, the decomposition method is particularly appropriate; it requires no perturbative or truncation methods or apriori assumptions of special behavior. Computer

results, on the other hand, are not correct when stochastic processes are discretized.

Supercomputers are developing rapidly because of urgent need in meteorology, fluid dynamics, fusion research, intelligent missile guidance, and weapons design. In fluid dynamics, for example, they are considered essential for the solution of the equations which are relevant to turbulence, internal waves in the ocean, and future development of hypersonic flight vehicles and engines. Supercomputers are also essential for VLSI devices, seismology, reservoir modeling, bioengineering, and studies of the national economy. The decomposition method now offers the possibility of substantial decrease in computational time over current supercomputers.

To solve, for example, the relevant problem of a space shuttle for eventual single-stage flight to a space station or servicing of space structures, a three-dimensional mesh is generated which discretizes the system of nonlinear partial differential equations into a million, a hundred million, or perhaps a billion coupled difference equations in as many unknowns. One begins to see then the tremendous data handling problem, the necessity for improved algorithms and the need for still greater computational speed. We may also have many unknowns at each point, and, as we have pointed out, the system nonlinearities and random fluctuations need to be taken into consideration. Since usually solutions are iterative - first solving an approximation to the original system of differential equations and then improving the solution by repeated substitution of each new solution - parallel processing is complicated by the difficulty of partitioning the work so each processor can work independently. This is being pursued by many ingenious ideas necessitated by the method of discretization.

In all such problems we need to be able to solve coupled systems of nonlinear (and generally stochastic as well) partial differential equations with complex boundary conditions and possible delayed effects.* These systems are *linearized* and *discretized* (and the stochastic aspects either ignored or improperly dealt with) so the various numerical approximation methods can be used. This requires faster and faster supercomputers to do these computations in a reasonable time. Fifth generation computers are being considered to operate at speeds up to 1000 megaflops (a gigaflop) or 10^9 operations per second or even higher.

Unfortunately the further developments in supercomputers can quite possibly still give wrong answers because even a single one-dimensional nonlinear differential equation without stochasticity in coefficients, inputs, and boundary conditions - let alone vector partial differential equations in space and time with nonlinear and/or stochastic parameters arising in control theory for space structures.

A supercomputer is, after all, a fast adding machine, and its computational accuracy is dependent on the sophistication of the mathematical methods programmed into it. Typical calculations consider millions of discrete time intervals made small enough so trajectories between them can be taken as low-order polynomials, e.g., quadratics. If stochasticity is involved, then Monte-Carlo methods are used, which insert randomness but not the properly correlated randomness which is present in the physical problem.

When one studies airflow about aircraft surfaces, computations are made at tens of millions of points, and it is felt that increasing the

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*Delayed effects are not discussed here but are dealt with in [2].

volume of computation to the limit in an ultimate extrapolation, super-computers will yield complete accuracy. Not only does this ignore stochasticity, it ignores the sensitivity of nonlinear stochastic systems to very slight changes in the model - in fact, to changes essentially undeterminable by measurement.

DECOMPOSITION TREATMENT OF NONLINEARITY COMPARED WITH LINEARIZATION

Nonlinear equations arise in every area of application, and the correct solution of dynamical systems modeled by nonlinear ordinary differential equations, systems of differential equations, partial differential equations, and systems of partial differential equations is vital to progress in many fields.

In order to make these equations tractable, it is quite common to linearize equations or assume "weak" nonlinearity, etc., because adequate methods simply have not been otherwise available. The practice of approximating a nonlinear function with a linearized version arose from the need to make equations tractable by simple analysis, since numerical solutions from computers can have drawbacks in that functional relationships are difficult to see in numerical printouts and, in many cases, computation can be excessive, and methods of analytical solution of nonlinear equations have been generally inadequate. It is known, of course, that the linearized solution can deviate considerably from the actual solution of the nonlinear problem and that linearization procedures require proof that the solution is valid. For example, writing $x'' = a \sin x$ in the form $x'' = ax$ requires a priori proof that x is sufficiently small. Usage of linearization has become rather standardized; however, solution of the actual nonlinear form is clearly preferable to a linear approximation.

The decomposition method has substantially improved our ability to solve a wide class of nonlinear and/or stochastic equations. It is now possible to obtain very accurate and verifiable solutions of nonlinear, or even nonlinear stochastic equations for all of the above types even if nonlinear, stochastic or coupled boundary conditions are involved.

Until now, linearization and perturbation have been essential procedures and one is reluctant to give up such a convenient analytical tool as superposition. However even with the increases in computer speed, the complexity of some of the problems demands new techniques to save computational time. This will be an important advantage of decomposition.

Exact linearization* is possible for some nonlinear equations so that a convenient check can be made of the decomposition solution. As an example, consider the nonlinear equation for $\phi(x,t)$

$$\phi_t + \phi_x + a\phi + \phi^2 = 0$$

with specified conditions. The transformation $\phi = 1/\psi(x,t)$ leads to the linear equation

$$\psi_t + \psi_x - \alpha\psi = 1$$

and conditions specified on ψ. This is now a linear equation which presents no difficulty if the conditions on ψ are specified. However, the nonlinear equation can be solved directly as follows. Using decomposition we write

$$L_t\phi + L_x\phi + a\phi + \phi^2 = 0$$

Let $N\phi = a\phi^2 = \sum_{n=0}^{\infty} A_n$ where

$$A_0 = \phi_0^2$$

$$A_1 = 2\phi_0\phi_1$$

*By exact linearization we refer to the cases where it is possible to use transformations of dependent and independent variables to transform a nonlinear equation into a linear equation.

$$A_2 = \phi_1^2 + 2\phi_0\phi_2$$

$$A_3 = 2\phi_1\phi_2 + 2\phi_0\phi_3$$

Solving for $L_t\phi$ and for $L_x\phi$

$$L_t\phi = -L_x\phi - a\phi - \phi^2$$

$$L_x\phi = -L_t\phi - a\phi - \phi^2$$

Thus

$$\phi = \phi(x,0) - L_t^{-1}L_x\phi - aL_t^{-1}\phi - L_t^{-1}\phi^2$$

$$\phi = \phi(0,t) - L_x^{-1}L_t\phi - aL_x^{-1}\phi - L_x^{-1}\phi^2$$

Adding

$$\phi = (1/2)\{\phi(x,0) + \phi(0,t) - (L_t^{-1}L_x + L_x^{-1}L_t)\phi$$

$$-a(L_t^{-1} + L_x^{-1})\phi + (L_t^{-1} + L_x^{-1})\phi^2\}$$

We define

$$\phi_0 = (1/2)\{\phi(x,0) + \phi(0,t)\}$$

substitute $\phi = \sum_{n=0}^{\infty} \phi_n$ and $\phi^2 = \sum_{n=0}^{\infty} A_n$ to obtain

$$\phi_{n+1} = -(1/2)\{L_t^{-1}L_x + L_x^{-1}L_t\}\phi_n$$

$$- (1/2)a\{L_t^{-1} + L_x^{-1}\}\phi_n - (1/2)\{L_t^{-1} + L_x^{-1}\}A_n$$

for $n \geq 0$ so that all components can be determined. The solution, of course, will be identical. However most equations cannot be exactly linearized by transformations, hence, the decomposition technique is much more useful.

In the analysis of systems it is common to suppose that everything is linear or sufficiently close to linear so that linearized analyses will be adequate. Thus in a mechanical system assumed to be linear, displacements and accelerations are proportional to forces. Thus we write output $y(t)$ as proportional to input $x(t)$ or $y = kx$ where k is a constant independent of t or x. Now suppose that the system deviates only slightly from linearity by adding a small term which is nonlinear, e.g., $y = kx + \epsilon x^2$. If we now consider the input $x = A \cos \omega_1 t + B \cos \omega_2 t$, we get not only the linear term

$$K[A \cos \omega_1 t + B \cos \omega_2 t]$$

but also

$$k\,[A^2 \cos^2 \omega_1 t + B^2 \cos^2 \omega_2 t + 2AB \cos \omega_1 t \cos \omega_2 t]$$

The first two of these produce constant terms and second harmonic terms as before. Also sum and difference frequencies arise from the cross product term

$$2AB \cos \omega_1 t \cos \omega_2 t = AB[\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t]$$

Thus, a nonlinear system produces new effects not present in linear systems: these effects are proportional to ϵ (and to products of amplitudes A^2, B^2, or AB). Clearly then, if ϵ is not small, such effects become important.

Solutions are generally carried out only under the assumption that ϵ is small so that perturbation theory will be applicable, i.e., when we consider a "slightly nonlinear" or a "weakly nonlinear" system.

Since, in general, solutions of nonlinear equations are made by linearizing the equations, it is natural to ask what the effect of

linearization is on the actual solutions. Let us consider the general nonlinear example

$$Ly + Ry + Ny = x(t)$$

where L is the invertible linear operator, R is the remaining linear operator, and Ny is the nonlinear term. We have

$$Ly = x - Ry - Ny$$

$$y = \phi + L^{-1}x - L^{-1}Ry - L^{-1}Ny$$

where $L\phi = 0$. Assume the solution is given by $y = \sum_{n=0}^{\infty} y_n$ with $y_0 = \phi + L^{-1}x$ identified as the first component of the sum. Assume also the decomposition of the nonlinear term Ny into $\sum_{n=0}^{\infty} A_n$ where the A_n are generated for the specific Ny. Then the components after y_0 are determinable in terms of y_0 as

$$y_1 = -L^{-1}Ry_0 - L^{-1}A_0(y_0)$$

$$y_2 = -L^{-1}Ry_1 - L^{-1}A_1(y_0,y_1)$$

$$y_3 = -L^{-1}Ry_2 - L^{-1}A_2(y_0,y_1,y_2)$$

$$\vdots$$

$$y_n = -L^{-1}Ry_{n-1} - L^{-1}A_{n-1}(y_0,\ldots,y_{n-1})$$

or, equivalently,

$$y_1 = (-L^{-1}R)y_0 - L^{-1}A_0$$

$$y_2 = (-L^{-1}R)^2y_0 - (-L^{-1}R)L^{-1}A_0 - L^{-1}A_1$$

$$y_3 = (-L^{-1}R)^3y_0 - (-L^{-1}R)^2L^{-1}A_0 - (-L^{-1}R)L^{-1}A_1 - L^{-1}A_2$$

$$\vdots$$

$$y_n = (-L^{-1}R)^n y_0 - \sum_{\nu=0}^{n-1} (-L^{-1}R)^{n-1-\nu} - L^{-1}A_\nu$$

for $n \geq 1$. The solution is $y = \sum_{n=0}^{\infty} y_n$ or

$$y = \sum_{n=0}^{\infty} (-L^{-1}R)^n y_0$$

$$- \sum_{n=1}^{\infty} \sum_{\nu=0}^{n-1} (-L^{-1}R)^{n-1-\nu} - L^{-1}A_\nu$$

It has been shown by Adomian that $A_n(y_0, y_1, \ldots, y_n)$ reduces to y_n if $Ny = f(y) = y$. Then the solution is

$$y = \sum_{n=0}^{\infty} (-L^{-1}R)^n y_0$$

$$- \sum_{n=1}^{\infty} \sum_{\nu=0}^{n-1} (-L^{-1}R)^{n-1-\nu} - L^{-1}y_\nu$$

i.e., the solution corresponds now to the equation $Ly + Ry + y = x$ which yields

$$y = y_0 - L^{-1}Ry - L^{-1}y$$

with

$$y_1 = -L^{-1}Ry_0 - L^{-1}y_0$$

$$y_2 = -L^{-1}Ry_1 - L^{-1}y_0$$

$$= (-L^{-1}R)^2 y_0 - (-L^{-1}R)L^{-1}y_0 - L^{-1}y_0$$

$$y_3 = -L^{-1}Ry_2 - L^{-1}y_1$$

$$= (-L^{-1}R)^3 y_0 - (-L^{-1}R)^2 L^{-1}y_0 - (-L^{-1}R)L^{-1}y_0 - L^{-1}y_1$$

$$\vdots$$

The result on the solution of replacing $Ny = f(y)$ by y can be determined by plotting the nonlinear and the linearized results for a specific $f(y)$ as we will see in the following example. Similarly replacing $f(y)$ by another perhaps more sophisticated linearization is seen by simply calculating the A_n for the linearized function replacing $f(y)$..

Example: Exponential nonlinearity: Let us consider a simple nonlinear ordinary differential equation with an exponential nonlinearity

$$dy/dx + e^y = 0$$

$$y(0) = 1$$

In our usual standard form (1983)* this is written

$$Ly + Ny = x$$

with $L = d/dx$ and $Ny = e^y$. We solve for Ly, i.e., $Ly = -Ny$ hen write $L^{-1}Ly = -L^{-1}Ny$ with L^{-1} defined as the integration over x. The left side is $y - y(0)$ hence

$$y = y(0) - L^{-1}Ny$$

The nonlinear term $Ny = e^y$ is replaced by $\sum_{n=0}^{\infty} A_n$ where the A_n can be written as $A_n(e^y)$ to emphasize that they are generated for this specific function. Thus,

$$y = y(0) - L^{-1} \sum_{n=0}^{\infty} A_n(e^y)$$

Now the decomposition of the solution y into $\sum_{n=0}^{\infty} y_n$ leads to the term-by-term identification

*See reference [1].

$$y_0 = y(0) = 1$$

$$y_1 = -L^{-1}A_0$$

$$y_2 = -L^{-1}A_1$$

$$\vdots$$

$$y_{n+1} = -L^{-1}A_n$$

The $A_n(e^y)$ are given by

$$A_0 = e^{y_0}$$

$$A_1 = y_1 e^{y_0}$$

$$A_2 = (y_1^2/2 + y_2)e^{y_0}$$

$$A_3 = (y_1^3/6 + y_1 y_2 + y_3)e^{y_0}$$

$$\vdots$$

Thus

$$y_0 = 1$$

$$y_1 = -ex$$

$$y_2 = e^2x^2/2$$

$$y_3 = -e^3x^3/3!$$

$$\vdots$$

$$y = 1 + \sum_{n=1}^{\infty} (-1)^n e^n x^n/n!$$

which can also be written $y = 1 - \ln[1 + ex]$ for $x < 1$.

Suppose we approximate the e^y term by $1 + y$, dropping all terms of the series for e^y except the constant and the linear term. The differential equation becomes

$$dy/dx = -(1 + y)$$

Then

$$y = y(0) - L^{-1}(1 + y)$$

$$= 1 - L^{-1}[1] - L^{-1} \sum_{n=0}^{\infty} y_n$$

so that

$$y_0 = 1 - x$$

$$y_1 = -L^{-1}(1 - x)$$

$$\vdots$$

$$y_n = -L^{-1}y_{n-1} \qquad (n \geq 1)$$

Since y is the sum of the components, we have

$$y = \sum_{n=0}^{\infty} \{(-1)^n x^n/n! + (-1)^{n+1} x^{n+1}/(n+1)!\}$$

$$y = e^{-x} + (e^{-x} - 1) = 2e^{-x} - 1$$

We will identify this linearized solution as y and compare with the solution y of the nonlinear equation. The results are given in the table

A COMPARISON OF ACTUAL AND LINEARIZED SOLUTIONS
(EXPONENTIAL NONLINEARITY)

x	Actual Solution y(x)	Linearized Solution $y_\ell(x)$
0	1.0000	1.0000
.1	.7595	.8097
.2	.5658	.6375
.3	.4036	.4816
.4	.2641	.3406
.5	.1417	.2131

Example (Anharmonic oscillator): As an example of a vibration problem, consider now the well-known anharmonic oscillator equation $d^2\theta/dt^2 + k^2 \sin\theta = 0$ for $\theta(0) = \gamma$ = constant and $\theta'(0) = 0$. Using the decomposition method, the solution is found to be

$$\theta(t) = \gamma - [(kt)^2/2!]\sin\gamma + [(kt)^4/4!]\sin\gamma\cos\gamma$$
$$- [(kt)^6/6!](\sin\gamma\cos^2\gamma - 3\sin^3\gamma] + \cdots$$

which becomes

$$\theta(t) = \gamma[1 - (k^2t^2/2!) + (kt)^4/4! - \cdots]$$

in the linearized case, i.e., for "small amplitude" motion which offers interesting comparison in order to determine when the smallness assumption is inappropriate because of large amplitude vibrations.

Example: Hyperbolic sine nonlinearity: Consider the equation

$$du/dt = k \sinh u/\alpha$$

where $u(0) = c > 0$ for $t > 0$.

If we assume that we can approximate $\sinh u/\alpha \simeq u/\alpha$, we have $du/dt - ku/\alpha = 0$. Now solving by decomposition with $u = \sum_{n=0}^{\infty} u_n$, $L = d/dt$, and L^{-1} as the definite integral from 0 to t

$$Lu - ku/\alpha = 0$$

$$L^{-1}Lu = L^{-1}(k/\alpha)u$$

$$u = u(0) + L^{-1}(k/\alpha)u = \sum_{n=0}^{\infty} u_n$$

$$u_0 = u(0) = c$$

$$u_1 = (k/\alpha)ct = ckt/\alpha$$

$$u_2 = c(kt/\alpha)^2/2!$$

$$u_3 = c(kt/\alpha)^3/3!$$

$$\vdots$$

$$u_m = c(kt/\alpha)^m/m!$$

$$\vdots$$

i.e., $u = ce^{kt/\alpha}$.

To solve the original equation with $\sinh u/\alpha$

$$u = \sum_{n=0}^{\infty} u_n = c + kL^{-1} \sum_{n=0}^{\infty} A_n$$

where the A_n are generated for $Nu = \sinh u/\alpha$. These are given by:

$$A_0 = \sinh(u_0/\alpha)$$

$$A_1 = (u_1/\alpha)\cosh(u_0/\alpha)$$

$$A_2 = (u_2/\alpha)\cosh(u_0/\alpha) + (1/2!)(u_1^2/\alpha^2)\sinh(u_0/\alpha)$$

$$A_3 = (u_3/\alpha)\cosh(u_0/\alpha) + (u_1/\alpha)(u_2/\alpha)\sinh(u_0/\alpha) + (1/3!)(u_1^3/\alpha^3)\cosh(u_0/\alpha)$$

$$A_4 = (u_4/\alpha)\cosh(u_0/\alpha) + [(1/2!)(u_2^2/\alpha^2) + (u_1/\alpha)(u_2/\alpha)]\sinh(u_0/\alpha) + (1/2!)(u_1^2/\alpha^2)(u_2/\alpha)\cosh(u_0/\alpha) + (1/4!)(u_1^4\alpha^4)\sinh(u_0/\alpha)$$

$$A_5 = (u_5/\alpha)\cosh(u_0/\alpha) + [(u_2/\alpha)(u_3/\alpha) + (u_1/\alpha)(u_4/\alpha)]\sinh(u_0/\alpha) + [(u_1/\alpha)(1/2!)(u_2^2/\alpha^2) + (1/2!)(u_1^2/\alpha^2)(u_3/\alpha)]\cosh(u_0/\alpha) + (1/3!)(u_1^3/\alpha^3)u_2/\alpha)\sinh(u_0/\alpha) + (1/5!)(u_1^5/\alpha^5)\cosh(u_0/\alpha)$$

$$A_6 = (u_6\alpha)\cosh(u_0/\alpha) + [(1/2!)(u_3^2/\alpha^2) + (u_2/\alpha)u_4/\alpha) + (u_1/\alpha)(u_5/\alpha)]\sinh(u_0/\alpha) + [(1/3!)(u_2^3/\alpha^3) + (u_1/\alpha)(u_2/\alpha)(u_3/\alpha)$$

$$+ (1/2!)(u_1^2/\alpha^2)(u_4/\alpha)] \cosh(u_0/\alpha)$$

$$+ [(1/2!)(u_1^2/\alpha^2)(1/2!)(u_2^2/\alpha^2)$$

$$+ (1/3!)(u_1^3/\alpha^3)(u_3/\alpha)] \sinh(u_0/\alpha)$$

$$+ (1/4!)(u_1^4/\alpha^4)(u_2/\alpha) \cosh(u_0/\alpha)$$

$$+ (1/6!)(u_1^6/\alpha^6) \sinh(u_0/\alpha)$$

$$\vdots$$

Now

$$u_0 = c$$

$$u_1 = kL^{-1}A_0 = kL^{-1}[\sinh u_0/\alpha] = kt \sinh c/\alpha$$

$$u_2 = kL^{-1}A_1 = kL^{-1}[(u_1/\alpha) \cosh(u_0/\alpha)]$$

$$= kL^{-1}[(kt/\alpha) \sinh(c/\alpha) \cosh(c/\alpha)]$$

$$= (k^2t^2/2!\alpha) \sinh(c/\alpha) \cosh(c/\alpha)$$

$$u_3 = kL^{-1}A_2 = kL^{-1}[(u_2/\alpha) \cosh(u_0/\alpha) + (1/2)(u_1^2/\alpha^2) \sinh(u_0/\alpha)]$$

$$= kL^{-1}[(k^2t^2/2)(1/\alpha)^2 \sinh(c/\alpha) \cosh^2(c/\alpha)$$

$$+ (1/2)(k^2t^2)(1/\alpha)^2 \sinh^3(c/\alpha)]$$

$$= (k^3t^3/3!)(1/\alpha)^2 \sinh(c/\alpha)[\sinh^2(c/\alpha) + \cosh^2(c/\alpha)]$$

$$u_4 = (k^4t^4/4!)(1/\alpha)^3[\sinh(c/\alpha) \cosh(c/\alpha)][5 \sinh^2(c/\alpha) + \cosh^2(c/\alpha)]$$

$$\vdots$$

thus, the correct solution is the sum of the u_n above while the linearized solution is

$$u = c[1 + kt/\alpha + (kt/\alpha)^2/2! + (kt/\alpha)^3/3! + \cdots]$$

If we assume $\alpha = c = k = 1$, we have the results $u = e^t$ in the linear case which can be compared with the nonlinear solution.

ERROR IN LINEARIZED SOLUTION
(FOR A HYPERBOLIC SINE NONLINEARITY)

t	Linearized Solution	Nonlinear Solution	% Error
0	1	1	0
.1	1.105170918	1.127402502	1.97%
.2	1.221402758	1.278624737	4.48%
.3	1.349858808	1.462380247	7.69%
.4	1.491824698	1.693614375	11.92%
.5	1.648721271	2.001468676	17.62%
.6	1.8221188	2.456234584	25.82%
.7	2.013752707	3.325545159	39.45%
.75	2.117000017	4.512775469	53.09%
.76	2.13827622	5.121285511	58.25%
.77	2.159766254	6.939848656	68.88%
.7719	2.163873711	10.9022661	80.15%
.771936	2.163951611	14.69149181	85.27%
.77193683	2.163953407	20.34929139	89.37%
.7719368329	2.163953414	28.3241683	92.36%
.7719368330	2.163953414	∞	100. %

So we see that the error due to linearization approaches 100% in this simple example. Linearization of nonlinearities in modelling of rapidly maneuvering space structures could conceivably result in similar inaccuracies and should be avoided.

Series Solution Convergence Compared with Numerical Integration: The decomposition method converges very rapidly in most cases: ϕ_n, the n-term approximation, is accurate for quite low values of n. To emphasize this point we now consider only a two-term approximation in the following example with the added comment that additional terms are extremely easy to obtain.

Consider the equation $dy/dt = t + y^{-1}$. We now have $L = d/dt$, $Ny = -y^{-1}$, $x(t) = t$. Assume $y(0) = k$ is an integer. Then

$$L^{-1}Ly = L^{-1}x - L^{-1}Ny$$

$$y - y(0) = L^{-1}t + L^{-1}\sum_{n=0}^{\infty} A_n$$

where the $A_n = A_n(y^{-1})$.

$$y = y(0) + L^{-1}\sum_{n=0}^{\infty} A_n$$

Let

$$y_0 = k + t^2/2$$

and since $A_0 = y_0^{-1}$, $A_1 = -y_0^{-2}y_1$, $A_2 = -y_0^{-2}y_2 + y_0^{-3}y_1^2, \ldots,$

$$y_1 = \int_0^t y_0^{-1}dt = \int_0^t (k + t^2/2)^{-1}dt$$

$$y_1 = (2/k)^{1/2}\tan^{-1}[t/(2k)^{1/2}]$$

$$\vdots$$

Let us consider a *two-term* approximation $\phi_2 = y_0 + y_1$. (The complete solution, of course, is $\sum_{n=0}^{\infty} y_n$.) Then

$$\phi_2 = k + t^2/2 + (2/k)^{1/2}\tan^{-1}[t/(2k)^{1/2}]$$

The following table compares this approximation with results of a numerical integration using $k = 4$. With only a little more effort we could go to a higher n in ϕ_n for a better approximation, which is necessary since the percentage error is already extremely small. The worst case is less than 0.4%. However, if we go to ϕ_3 we find the worst case has an error less than 0.02% - this for only a three term approximation! Thus, we have very rapid convergence.

COMPARISON OF DECOMPOSITION AND NUMERICAL INTEGRATION

t	Decomposition method ϕ_1	Numerical integration y	$\Delta = \phi_2 - y$	% Error
0	4.0	4.0	0	0
0.5	4.25	4.25	0	0
1.0	4.74	4.73	0.01	0.21
1.5	5.47	5.46	0.01	0.18
2.0	6.44	6.42	0.02	0.31
2.5	7.64	7.61	0.03	0.39
3.0	9.08	9.05	0.03	0.33
4.0	12.68	12.64	0.04	0.32
5.0	17.25	17.21	0.04	0.23
10.0	54.92	54.88	0.04	0.07
20.0	205.01	204.97	0.04	0.02

HEAT AND DIFFUSION

Typical problems of this type are initial-boundary value problems over a specified region and with a given initial temperature. Let's look at some common equations involving nonlinear terms which may arise in a particular situation - all of which are easily dealt with.

Burger's equation $u_t + uu_x = u_{xx}$ represents a balance between diffusion and convection. If the u_{xx} term vanishes, we have a simple diffusion equation - a scalar hyperbolic equation $u_t = u_{xx}$. Some specific conditions must be specified for completeness, e.g.,

$$u(x,0) = f(x) \qquad \text{for } -\infty < x < \infty$$

$$u(0,t) = 0$$

and

$$u_x(0,t) = h(t)$$

Another possibility is

$$u_t + uu_x + \lambda u = \alpha u_{xx} \qquad \lambda > 0$$

or the inhomogeneous Burger's equation

$$u_t + uu_x = \alpha u_{xx} + f(x,t)$$

for $0 \leq x \leq \ell$ and $0 \leq t \leq T$ with $u(0,t) = u(L,t) = 0$ and $u(x,0) = a(x)$ where $a(0) = a(L) = 0$ and $\alpha > 0$.

Fisher's equation $u_t = u_{xx} + u(1 - u)$ shows still another nonlinearity $u(1-u)$.

For a general nonlinear form, we can consider

$$(\partial/\partial x)[k(u)\partial u/\partial x] = c\rho\partial u/\partial t$$

with $c\rho$ a constant and $k(u)$ a specified function. If $k = 1$, this becomes the simple heat equation $u_{xx} = u_t$. When temperature variations increase, one must consider k dependent on temperature u, that is, $k = k(u)$ so that the equation becomes nonlinear. Suppose, for example, $k = k_0 + k_1 u$, etc. Finally, we can have equations such as $u_t = u_{xx} + \partial f(u,u_x)/\partial t$ or $u_t = a^2 u_{xx} + f(x,t)$, etc. All of the foregoing are easily dealt with by decomposition with the A_n polynomials generated for the particular nonlinear terms.

SOME SPECIAL CASES DEPENDENT ON STRUCTURE AND MATERIALS

1) One-dimensional heat equation: Consider a single space dimension with a specified temperature distribution $u(x,t)$ described by

$$(\partial/\partial x)(k\ \partial u/\partial x) + F(x,t) = c\rho(\partial u/\partial t)$$

where F represents a heat source at x at time t. In the mathematically simplest case where k, c, ρ are assumed to be constants, we can write this in the form

$$u_t = a^2 u_{xx} + f(x,t)$$

where $a^2 = k/c\rho$ and $f(x,t) = (1/c\rho)F(x,t)$ or if the source vanishes, simply

$$u_t = a^2 u_{xx}$$

often referred to as the heat conduction equation. We can consider, for example, a homogeneous rod of length ℓ, thermally insulated on lateral surfaces and sufficiently thin so that at any time t, the temperature $u(x,t)$ is the same at all points of the cross-section at x.

If the rod is inhomogeneous so that $k = k(x)$ and heat exchange takes place with the surrounding medium, the situation is described by an equation of the form

$$u_t - a^2 u_{xx} + \alpha u = f(x,t)$$

with $\alpha = h/c\rho$, h a heat-exchange coefficient, and $f(x,t) = \alpha T(x,t) + g(x,t)/c\rho$ with T the temperature of the medium and $g(x,t)$ the density of the heat sources.

2) *Three-dimensional heat equation:* In the case of three space dimensions and time, the heat flow is described by $u(x,y,z,t)$ and the equation

$$c\rho u_t = \text{div}(k \text{ grad } u) + F$$

where k, c, ρ are functions of x, y, z. More generally we may have k dependent on u and space coordinates, e.g., in the case of large temperature fluctuation. This is the more physically realistic case which we can apply to real trusses and completely amenable to solution by the decomposition method.

If the material is homogeneous, we can write

$$u_t = a^2(u_{xx} + u_{yy} + u_{zz}) + F/c\rho$$

where $a^2 = k/c\rho$, or,

$$u_t = a^2\nabla^2 u + f \qquad f = F/c\rho.$$

The diffusion equation is analogous to the heat conduction equation. For one space dimension and time, we can write

$$\partial/\partial x(D\ \partial u/\partial x) = c(\partial u/\partial t)$$

or $(Du_x)_x = cu_t$. If the diffusion coefficient D is constant, we have

$$u_t = a^2 u_{xx}$$

in the same form as before with $a^2 = D/c$.

3) Boundary conditions for Heat problems: Differential equations do not completely specify solution uniquely. We must also prescribe conditions on the solution. Thus to complete the specification of the problem, we require the auxiliary initial and/or boundary conditions as well. Thus in the heat equation, we can prescribe the temperature at the ends $x = 0$ and $x = \ell$. We can specify $u(0,t) = g(t)$ where $g(t)$ is known in the time interval of interest. Or, we can specify the heat flow at one end, e.g., $\partial u(0,t)/\partial x = h(t)$. Or, we can have more complicated conditions such as

$$\partial u(\ell,t)/\partial x + \lambda u(\ell,t) = T(t)$$

These restrictions apply to all partial differential equations. Such equations have many solutions. In order to identify a particular solution from the entire possible set of solutions, we need supplementary conditions similar to specification of initial conditions in an initial-value problem described by an ordinary differential equation.

The usual situation is that we will have some boundary conditions specified on the boundaries of the space described by the coordinates x_1, x_2, x_3 or x, y, z for which the partial differential equation is

applicable and, also, initial conditions at a particular instant $t = 0$ or $t = t_0$.

If we know the temperature distribution at $t = 0$ and the temperature at the boundaries, the temperature distribution will be precisely determined for $t > 0$. If the time interval of interest is very long, the influence of the initial condition vanishes and the distribution will be determined by the boundary conditions. For $0 \leq x \leq \ell$, we might specify $u(0,t) = f(t)$ and $u(\ell,t) = g(t)$. For $0 < x < \infty$, we need only $u(0,t) = f(t)$.

If the region is unbounded, then a bounded solution is uniquely determined by the initial conditions alone, i.e., by $u(0,u_1,x_2,x_3)$. Thus for the bounded case we might have conditions such as u at $t = 0$ and u at $x = 0$ and $x = a$.

However, if for very large a, we retain the conditions at $t = 0$ and $x = 0$, but now $u(a)$ approaches zero as a becomes very large.

Use of the given conditions is simple. Thus in a second-order differential equation with no forcing term, i.e., one involving d^2u/dx^2, our u_0 term in the decomposition $u = \sum_{n=0}^{\infty} u_n$ is given by $u_0 = \gamma_1 + \gamma_2 x$. If the temperature u is given at the ends $x = 0$ and $x = a$ of the bar, we can evaluate γ_1 and γ_2 in terms of the given temperatures.

4) Solution of a Two-dimensional heat-flow equation: Consider the parabolic equation $u_{xx} + u_{yy} = ku_t$ where $k = k(x,y)$. Write this as

$$L_{xx}u + L_{yy}u = k(x,y)L_t u$$

We write three equations

$$L_{xx}u = kL_t u - L_{yy}u$$

$$L_{yy}u = kL_t u - L_{xx}u$$

$$L_t u = k^{-1}[L_{xx} + L_{yy}]u$$

Operate on the first with L_{xx}^{-1}, the second L_{yy}^{-1}, and the third with L_t^{-1} to get

$$u = u(0,y,t) + \gamma_1 x + L_{xx}^{-1}kL_t u - L_{xx}^{-1}L_{yy}u$$

$$u = u(x,0,t) + \gamma_2 y + L_{yy}^{-1}kL_t u - L_{yy}^{-1}L_{xx}u$$

$$u = u(x,y,0) + L_t^{-1}k^{-1}[L_{xx} + L_{yy}]u$$

Adding and dividing by three

$$u = (1/3)\{u(0,y,t) + u(x,0,t) + u(x,y,0)\}$$

$$+ (1/3)\{\gamma_1 x + \gamma_2 y\}$$

$$+ (1/3)\{[L_{xx}^{-1} + L_{yy}^{-1}]kL_t u$$

$$- [L_{xx}^{-1}L_{yy} + L_{yy}^{-1}L_{xx}]u$$

$$+ L_t^{-1}k^{-1}[L_{xx} + L_{yy}]u\}$$

Now u is replaced by $\sum_{n-0}^{\infty} u_n$ with u_0 identified as

$$u_0 = (1/3)\{u(0,y,t) + u(x,0,t)$$

$$+ u(x,y,0) + \gamma_1 x + \gamma_2 y\}$$

The boundary conditions at $x = a$ and $y = b$ determine γ_1, γ_2. These vanish if a and b recede to infinity. Each u_n after u_0 is now determined by:

$$u_n = (1/3)\{[L_{xx}^{-1} + L_{yy}^{-1}]kL_t u_{n-1}$$

$$- [L_{xx}^{-1}L_{yy} + L_{yy}^{-1}L_{xx}]u_{n-1}$$

$$+ L_t^{-1}k^{-1}[L_{xx} + L_{yy}]u_{n-1}$$

L_{xx} involves two differentiations with respect to x. L_{yy} involves two differentiations with respect to y. L_t involves a single differentiation with respect to t. The inverses are the corresponding integrations.

Thus, all the components of $u = \sum_{n=0}^{\infty} u_n$ are determinable and again we see that the decomposition method solves such equations.

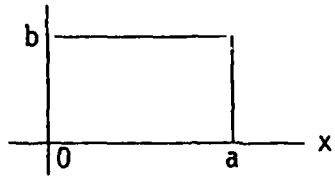
5) Solution of a two-dimensional heat-flow equation with a heat source g: We will consider a bounded rectangular plate with temperature specified on the edges and at an initial time $t = 0$. The equation to be considered is

$$u_{xx} + u_{yy} - k(x,y)u_t = g$$

or equivalently

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - k(x,y)\ \partial u/\partial t = g$$

for $0 < t < \infty$ and $x,y \in \Omega$, a bounded rectangular plate with boundary Γ with temperature distribution specified on the edge $x \in [0,a]$, and on $y \in [0,b]$, $k = k(x,y)$, given $u(x,y,0)|_{\Omega} = 0$ and $u(x,y,t) = f(x,y,t)$ for $x,y \in \Gamma$.



Suppose now we let the conductivity k be constant and the temperature T be constant on the given surfaces. Then

$$u_0 = T - (Tx/a + Ty/b)/3$$

If $a,b \to \infty$, then $u_0 = T$, i.e., u_0 approaches T for large a,b

$$u_1 = 1/3[L_{xx}^{-1} + L_{yy}^{-1}]u_0 - 1/3[L_{xx}^{-1}L_{yy} + L_{yy}^{-1}L_{xx}]u_0$$

$$+ 1/3\, L_t^{-1}k^{-1}[L_{xx} + L_{yy}]u_0 = (T/6)(x^2 + y^2)$$

We know u_0, the first component of $u = \sum_{n=0}^{\infty} u_n$ and we have now found u_1 in terms of u_0. All other components are found in the same way. If we replace u_1 by u_{n+1} and u_0 by u_n and let $n = 0$, we have the above formula. If we let $n = 1$, we have u_2 in terms of u_1. Then let $n = 2,3,\ldots$ to get others.

6) Three-dimensional general case: Let us consider the general form where k may be a constant, space or time dependent, or a function of u. In the last case, we have a nonlinear equation. The nonlinear terms must be replaced by the sum of the appropriate A_n as specified by the decomposition method

$$\sum_{i=1}^{3} (\partial/\partial x_i)(k\ \partial u/\partial x_i) - c\rho\partial u/\partial t = g$$

where k is the coefficient of thermal conductivity, ρ is the density, and c is the specific heat at x_1, x_2, x_3. We will use x, y, z to avoid confusion with the subscripts for the decomposition. Let $L_x = \partial/\partial x$, $L_y = \partial/\partial y$, $L_z = \partial/\partial z$, $L_t = \partial/\partial t$. Now we have

$$L_x k L_x u + L_y k L_y u + L_z k L_z u - c\rho L_t u = g$$

Solving for each term in turn,

$$L_x k L_x u = g + c\rho L_y u - L_y k L_y u - L_z k L_z u$$

$$L_y k L_y u = g + c\rho L_t u - L_x k L_x u - L_z k L_z u$$

$$L_z k L_z u = g + c\rho L_t u - L_x k L_x u - L_y k L_y u$$

$$c\rho L_t u = -g + L_x k L_x u + L_y k L_y u + L_z k L_z u$$

The first equation becomes

$$k L_x u = u(0,y,z,t) + L_x^{-1} g + L_x^{-1} c\rho L_t u$$

$$- L_x^{-1} L_y k L_y u - L_x^{-1} L_z k L_z u$$

$$L_x u = k^{-1}[u(0,y,z,t) + L_x^{-1} g]$$

$$+ k^{-1} L_x^{-1} c\rho L_t u - k^{-1} L_x^{-1} L_y k L_y u$$

$$- k^{-1} L_x^{-1} L_z k L_z u$$

$$u = u(0,y,z,t) + L_x^{-1}\{k^{-1}[u(0,y,z,t) + L_y^{-1} g]\}$$

$$+ L_x^{-1} k^{-1} L_x c\rho L_t u - L_x^{-1} k^{-1} L_x^{-1} L_y k L_y u$$

$$- L_x^{-1} k^{-1} L_x^{-1} L_z k L_z u$$

We proceed similarly with the other three equations. Then

$$k L_y u = u(x,0,z,t) + L_y^{-1} g + L_y^{-1} c\rho L_t u - L_y^{-1} L_x k L_x u - L_y^{-1} L_z k L_z u$$

$$L_y u = k^{-1}[u(x,0,z,t) + L_y^{-1} g] + k^{-1} L_y^{-1} c\rho L_t u - k^{-1} L_y^{-1} L_x k L_x u$$

$$- k^{-1} L_y^{-1} L_z k L_z u$$

$$u = u(x,0,z,t) + L_y^{-1} k^{-1}[u(x,0,z,t) + L_y^{-1} g]$$

$$+ L_y^{-1} k^{-1} L_y^{-1} c\rho L_t u - L_y^{-1} k^{-1} L_y^{-1} L_x k L_x u$$

The third equation becomes

$$k L_z u = u(x,y,0,t) + L_z^{-1} g + L_z^{-1} c\rho L_t u$$

$$- L_z^{-1} L_x k L_x u - L_z^{-1} L_y k L_y u$$

$$L_z u = k^{-1} u(x,y,0,t) + k^{-1} L_z^{-1} g + k^{-1} L_z^{-1} c\rho L_t u$$

$$- k^{-1} L_z^{-1} L_x k L_x u - k^{-1} L_z^{-1} L_y k L_y u$$

$$u = u(x,y,0,t) + L_z^{-1}k^{-1}u(x,y,0,t) + L_z^{-1}k^{-1}L_z^{-1}g$$

$$+ L_z^{-1}k^{-1}L_z^{-1}c\rho L_t u - L_z^{-1}k^{-1}L_z^{-1}L_x kL_x u$$

$$- L_z^{-1}k^{-1}L_z^{-1}L_y kL_y u$$

The fourth equation is

$$L_t u = -g/c\rho + (1/c\rho)(L_x kL_x u) + (1/c\rho)(L_y kL_y u)$$

$$+ 1/c\rho(L_z kL_z u)$$

$$u = u(x,y,z,0) - L_t^{-1}(g/c\rho) + L_t^{-1}(1/c\rho)L_x kL_x u$$

$$+ L_t^{-1}(1/c\rho)(L_y kL_y u)$$

Adding and dividing by four we get u:

$$u(0,y,z,t) + L_x^{-1}\{k^{-1}[u(0,y,z,t) + L_y^{-1}g]\}$$

$$+ L_x^{-1}k^{-1}L_x c\rho L_t u - L_x^{-1}k^{-1}L_x^{-1}L_y kL_y u$$

$$- L_x^{-1}k^{-1}L_x^{-1}L_z kL_z u$$

$$+ u(x,0,z,t) + L_y^{-1}k^{-1}[u(x,0,z,t) + L_y^{-1}g]$$

$$+ L_y^{-1}k^{-1}L_y^{-1}c\rho L_t u - L_y^{-1}k^{-1}L_y^{-1}L_x kL_x u$$

$$+ u(x,y,0,t) + L_z^{-1}k^{-1}u(x,y,0,t) + L_z^{-1}k^{-1}L_z^{-1}g$$

$$+ L_z^{-1}k^{-1}L_z^{-1}c\rho L_t u - L_z^{-1}k^{-1}L_z^{-1}L_x kL_x u$$

$$- L_z^{-1}k^{-1}L_z^{-1}L_y kL_y u$$

$$+ u(x,y,z,0) - L_t^{-1}(g/c\rho) + L_t^{-1}(1/c\rho)L_x k L_x u$$

$$+ L_t^{-1}(1/c\rho)(L_y k L_y u)$$

Replacing u by $\sum_{n=0}^{\infty} u_n$ or $u_0 + u_1 + \cdots$, and defining u_0 to include all the auxiliary condition terms and terms involving g, we have

$$u_0 = (1/4)\{u(0,y,z,t) + u(x,0,z,t) + u(x,y,0,t)$$

$$+ u(x,y,z,0) + L_x^{-1}k^{-1}u(0,y,z,t)$$

$$+ L_y^{-1}k^{-1}u(x,0,z,t) + L_z^{-1}k^{-1}u(x,y,0,t)$$

$$- L_t^{-1}(g/c\rho) + L_z^{-1}k^{-1}L_z^{-1}g + L_y^{-1}k^{-1}L_y^{-1}g$$

$$+ L_x^{-1}k^{-1}L_x^{-1}g\}$$

Now all remaining terms $u_1, u_2, \ldots,$ can be found from preceding terms by:

$$u_{n+1} = (1/4)\{L_x^{-1}k^{-1}L_x^{-1}c\rho L_t - L_x^{-1}k^{-1}L_x^{-1}kL_y$$

$$- L_x^{-1}k^{-1}L_x^{-1}L_z kL_z + L_y^{-1}k^{-1}L_y^{-1}c\rho L_t$$

$$- L_y^{-1}k^{-1}L_y^{-1}L_x kL_x + L_z^{-1}k^{-1}L_z^{-1}c\rho L_t$$

$$- L_z^{-1}k^{-1}L_z^{-1}L_x kL_x - L_z^{-1}k^{-1}L_z L_y kL_y$$

$$+ L_t^{-1}(1/c\rho)L_x kL_x + L_t^{-1}(1/c\rho)L_y kL_y\}u_n$$

for $n \geq 0$.

7) Three-dimensional linear case: Of course we are by no means limited to linear equations; the decomposition method solves highly nonlinear equations. However we consider a case familiar to engineers for clarity and simplicity. Now assuming k, c, ρ are constants, we have the form:

$$a^2\nabla^2 u - u_t = f$$

Now define $L_{xx} = \partial^2/\partial x^2$, $L_{yy} = \partial^2/\partial y^2$, $L_{zz} = \partial^2/\partial z^2$, $L_t = \partial/\partial t$. Then

$$a^2[L_{xx} + L_{yy} + L_{zz}]u - L_t u = f$$

Solving for each linear operator term in turn, we have the four equations

$$a^2 L_{xx} u = f + L_t u - a^2 L_{yy} u - a^2 L_{zz} u$$

$$a^2 L_{yy} u = f + L_t u - a^2 L_{xx} u - a^2 L_{zz} u$$

$$a^2 L_{zz} u = f + L_t u - a^2 L_{xx} u - a^2 L_{yy} u$$

$$L_t u = -f + a^2 L_{xx} u + a^2 L_{yy} u + a^2 L_{zz} u$$

or

$$L_{xx} u = a^{-2} f + a^{-2} L_t u - L_{yy} u - L_{zz} u$$

$$L_{yy} u = a^{-2} f + a^{-2} L_t u - L_{xx} u - L_{zz} u$$

$$L_{zz} u = a^{-2} f + a^{-2} L_t u - L_{xx} u - L_{yy} u$$

$$L_t u = -f + a^2 L_{xx} u + a^2 L_{yy} u + a^2 L_{zz} u$$

For simplicity let us allow $a = 1$ here. Then

$$L_{xx} u = f + L_t u - L_{yy} u - L_{zz} u$$

$$L_{yy} u = f + L_t u - L_{xx} u - L_{zz} u$$

$$L_{zz}u = f + L_t u - L_{xx}u - L_{yy}u$$

$$L_t u = -f + L_{xx}u - L_{yy}u - L_{zz}u$$

Operating with the inverse operators (two-fold integration for x, y, z and single for t),

$$u(x,y,z,t) = A + Bx + L_{xx}^{-1}f + [L_{xx}^{-1}L_t - L_{xx}^{-1}L_{yy} - L_{xx}^{-1}L_{zz}]u$$

$$u(x,y,z,t) = C + Dy + L_{yy}^{-1}f + [L_{yy}^{-1}L_t - L_{yy}^{-1}L_{xx} - L_{yy}^{-1}L_{zz}]u$$

$$u(x,y,z,t) = E + Fz + L_{zz}^{-1}f + [L_{zz}^{-1}L_t - L_{zz}^{-1}L_{xx} - L_{zz}^{-1}L_{yy}]u$$

$$u(x,y,z,t) = G - L_t^{-1}f + [L_t^{-1}L_{xx} + L_t^{-1}L_{yy} + L_t^{-1}L_{zz}]u$$

Again, adding these equations and dividing by four, we have u on the left side and a complex expression on the right side. Letting $u = \sum_{n=0}^{\infty} u_n$ we identify u_0 as including terms involving the boundary conditions and the forcing function f. $A, B, \ldots, G$ are chosen to satisfy boundary conditions. Now we have

$$u_0 = (1/4)\{A + Bx + C + Dy + E + Fz + G$$

$$- [L_{xx}^{-1} + L_{yy}^{-1} + L_{zz}^{-1} - L_t^{-1}]f\}$$

$$u_{n+1} = (1/4)\, L_{xx}^{-1}L_t - L_{xx}^{-1}L_{yy} - L_{xx}^{-1}L_{zz} + L_{yy}^{-1}\, L_t - L_{yy}^{-1}L_{xx} - L_{yy}^{-1}L_{zz}$$

$$+ L_{zz}^{-1}L_t - L_{zz}^{-1}L_{xx} - L_{zz}^{-1}L_{yy} + L_t^{-1}L_{xx} + L_t^{-1}L_{yy} + L_t^{-1}L_{zz}\}u_n$$

for $n \geq 0$. The expression $\phi_n = \sum_{i=0}^{n-1} u_i$ is the approximation to u, i.e., the correct solution is the sum of all the components u_i but since the series converges so rapidly that a few terms are sufficient for a good

approximation, some n terms will be enough or $\phi_n = u_0 + u_1 + \cdots + u^{n-1}$ or $\sum_{i=0}^{n-1} u_i$ will serve as an approximation to u. Easily computable and accurate solutions have been obtained with small n for differential and partial differential equations, even when nonlinear terms or stochastic processes are included. Numerical computations clearly demonstrate the convergence to desired accuracy.

8) Two-dimensional homogeneous heat equation: As an illustration of procedure, we consider the specific example

$$\nabla^2 u - u_t = 0$$

with given conditions $u(0,t) = t$, $u(x,0) = x^2/2$, and $\partial u/\partial x|_{x=0} = 0$; we now have

$$u = u(0,t) + L_{xx}^{-1} L_t u$$

$$u = u(x,0) + L_t^{-1} L_{xx} u$$

Adding and dividing by two, we have

$$u = (1/2)[u(0,t) + u(x,0)] + (1/2)[L_{xx}^{-1} L_t + L_t^{-1} L_{xx}] \sum_{n=0}^{\infty} u_n$$

where we have written $u = \sum_{n=0}^{\infty} u_n$. Now we can identify

$$u_0 = (1/2)[t + (x^2/2)]$$

$$u_1 = (1/2)[L_{xx}^{-1} L_t + L_t^{-1} L_{xx}] u_0$$

$$u_2 = (1/2)[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}]u_1$$

$$\vdots$$

$$u_{n+1} = (1/2)[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}]u_n \qquad n \geq 0$$

This specific case with the given conditions has been chosen so that $[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}]$ is an identity operator, a special case making the calculation particularly simple for the sake of clarity. Thus we see that

$$[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}][u(0,t) + u(x,0)]$$

$$= L_{xx}^{-1}L_t u(0,t) + L_{xx}^{-1}L_t u(x,0) + L_t^{-1}L_{xx}u(0,t) + L_t^{-1}L_{xx}u(x,0)$$

$$= L_{xx}^{-1}L_t(t) + L_{xx}^{-1}L_t(x^2/2) + L_t^{-1}L_{xx}(t) + L_t^{-1}L_{xx}(x^2/2)$$

$$= (x^2/2) + t = u(x,0) + u(0,t)$$

In the general case one has to compute the effects of these operators repeatedly, but it is still straightforward. Now for the case being considered

$$u_0 = (1/2)[t + (x^2/2)]$$

$$u_1 = (1/2)[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}]u_0 = (1/4)[t + (x^2/2)]$$

$$u_2 = (1/2)[L_{xx}^{-1}L_t + L_t^{-1}L_{xx}]u_1 = (1/8)[t + (x^2/2)]$$

$$\vdots$$

$$u_n = (1/2^{n+1})[t + (x^2/2)]$$

$$u = [t + (x^2/2)] \sum_{n=0}^{\infty} (1/2^{n+1})$$

is the solution obtained from summing the u_n from 0 to ∞. Denoting

an n-term approximation by ϕ_n,

$$\phi_1 = 0.5(t + x^2/2) = u_0$$

$$\phi_2 = 0.75(t + x^2/2) = u_0 + u_1$$

$$\phi_3 = 0.875(t + x^2/2) = u_0 + u_1 + u_2$$

$$\vdots$$

$$u = \lim_{n\to\infty} \phi_n = u_1 + u_1 + \cdots + u_{n-1} = (1)(t + x^2/2)$$

which shows the improving approximations to the correct solution $t + x^2/2$ as n increases. We see the convergence of the coefficients .5, 0.75, 0.875... to 1.0 illustrating the improving approximation. Solutions using the decomposition method are analytic *but not* closed form - the closed form in this case arose from a problem chosen for the purpose of clear illustration. Note that six terms yield the solution to better than 98% and with 10 terms, the approximation is within 99.9% of the correct value.

Addition of stochasticity in forcing functions or coefficients results in the series ϕ_n becoming a stochastic series from which statistics are easily obtained by averaging.

Example: Heat conduction in a uniform beam: For one-dimensional flow parallel to the x-axis, the equation is $u_t = ku_{xx}$ where k is the thermal diffusivity. Taking $k = 1$, we have $u_t = u_{xx}$ if heat is transferred only by conduction, and no internal source is present.

Boundary conditions describing the thermal conditions on the surface of the solid and the initial temperature distribution are necessary along

with the heat equation to determine the temperature distribution $u(x,t)$. (For a three-dimensional solid with an insulated end at $x = 0$, the condition on u is $u_x(0,y,z,t) = 0$ because the heat flow across the surface would be proportional to $u_x(0,y,z,t)$.)

Suppose we have a bar whose temperature distribution is described by $u(x,t)$. Let the bar be initially at temperature $f(x) = \sin x$ and assume the ends $x = 0$ and $x = \pi$ are kept at temperature zero. We have therefore the conditions $u(x,0) = \sin x$, and $u(0,t) = u(\pi,t) = 0$. Writing $L_{xx}u = L_t u$ and applying the inverse operator L_{xx}^{-1} to both sides, we get

$$u = A + Bx + L_{xx}^{-1}L_t u.$$

A and B must be zero to satisfy the given boundary conditions at the ends so that u_0 is zero in this equation. Whenever this happens, that particular equation does not contribute to the solution. (The decomposition method requires a non-zero u_0.) Now applying L_t^{-1} to $L_t u = L_{xx}u$ and satisfying the initial condition, we have

$$u = u(x,0) + L_t^{-1}L_{xx}u$$

so that $u_0 = u(x,0) = \sin x$. We have

$$u = u_0 + L_t^{-1}L_{xx}\sum_{n=0}^{\infty} u_n$$

or

$$u_1 = L_t^{-1}L_{xx}u_0 = L_t^{-1}L_{xx}\sin x = -t\sin x$$

$$u_2 = L_t^{-1}L_{xx}u_1 = L_t^{-1}L_{xx}(-t\sin x) = (t^2/2)\sin x$$

$$\vdots$$

The solution is $u = (1 - t + t^2/2 - \ldots) \sin x = e^{-t} \sin x$ which is easily verified.

It is interesting, as a simple test, also to consider the trivial example of steady-state temperature in a bar with ends at $x = 0$ and $x = 1$, maintained at $u = 0$ and $u = T$ respectively. Then $u(0) = 0$ and $u(1) = T$. Since we are considering steady state, we have no time derivative and only $u_{xx} = 0$ so that $u = Ax + B$ is the complete solution instead of just the first component u_0. Applying the conditions, i.e., the known temperatures at the ends, we have $B = 0$ and $A = T$ so that $u = Tx$ is the well-known solution.

9) Conductivity dependent on x: Consider a rod of length ℓ lying along the x axis with its left end at the origin. Suppose its density is a function of x, its cross-section is A, its mass per unit length is ρ, specific heat is c, and conductivity is k_1. Let $u(x,t)$ represent temperature at a point x at time t and let $F(x,t)$ be the amount of heat per unit cross-section per unit time passing x to the right. Then, the cylinder of base area A between the points x and $x + \delta$ has heat supplied to it at the rate $c\rho\delta Au_t = AF(x,t) - AF(x + \delta,t) = -AF_x\delta$ where higher powers of δ are dropped since δ is small. Since $F = -k_1u_x$ in heat conduction, $F_x = -k_1u_{xx}$. Now we have the model equation

$$(c\rho/k_1)u_t = u_{xx}$$

or letting $k = c\rho/k_1$, we have the heat equation

$$u_{xx} = k(x)u_t$$

where conductivity k is not a constant but is dependent on position. Suppose we have initial boundary conditions $u(x,0) = g(x)$ for $0 < x < 1$ and $u(0,t) = h_1(t)$, $u(1,t) = h_2(t)$ for $t > 0$. Then

$$u_0 = (1/2)[g(t) + h_1(t) + x(h_2(t) - h_1(t))]$$

from which the following terms can be computed analogous to the previous procedure when k was assumed constant. However we now have

$$L_{xx}u = k(x)L_t u$$

or

$$u = A + Bx + L_{xx}^{-1}k(x)L_t u, \qquad u_0 = A + Bx$$

and

$$k(x)L_t u = L_{xx}u$$

$$L_t u = k^{-1}(x)L_{xx}u$$

$$u = C + L_t^{-1}k^{-1}(x)L_{xx}u, \qquad u_0 = c$$

so that if neither u_0 vanishes

$$u = (1/2)[A + Bx + C] + (1/2)[L_{xx}^{-1}k(x)L_t + L_t^{-1}k^{-1}(x)L_{xx}]u$$

$$u_0 = (1/2)[A + Bx + C]$$

$$u_{n+1} = (1/2)[L_{xx}^{-1}k(x)L_t + L_t^{-1}k^{-1}(x)L_{xx}]u_n$$

Now we can consider various specific examples. Our only objective here is to demonstrate solutions of problems also solvable by other methods for easy verification and to show the method clearly and the simplicity of

the computation involved. Let $h_1 = h_2 = 0$ and $g(x) = \sin \pi x$. Then one equation is sufficient. If $k = 1$,

$$u_0 = \sin \pi x$$

$$u_1 = L_t^{-1} L_{xx} \sin \pi x = -\pi^2 t \sin \pi x$$

$$u_2 = L_t^{-1} L_{xx} (-\pi^2 t \sin \pi x) = (\pi^4 t^2/2) \sin \pi x$$

$$\vdots$$

$$u = [1 - (\pi^2 t) + (\pi^4 t^2/2) - \cdots] \sin \pi x$$

At this point we happen to be able to identify the bracketed series as $e^{-\pi^2 t}$ so we can write

$$u = e^{-\pi^2 t} \sin \pi x$$

Identification of a closed form is not necessary. We have an analytic approximation that can be calculated numerically to see it converges to the correct result. Thus the temperature $u(x,t)$ can be calculated exactly at any x and t. To consider more general cases we can allow a given $k(x)$, e.g., $k(x) = 1 + x$, and compute a series for $u(x,t)$. The method can now be generalized to parabolic equations such as

$$\nabla^2 u - k u_t = g$$

where $k = k(x,y)$ or $k(x,y,z)$ and $g = g(x,y,z,t)$ or even stochastic cases.

10) Nonlinear heat equation in one-dimension:

$$(\partial/\partial x)[k(u)\partial u/\partial x] = c\rho(\partial u/\partial t)$$

$$u(0,t) = u_1$$

$$u(x,0) = u_2$$

Here, we can let $k(u)$ be any function - a linear function $k_0 + k_1 u$, or $\alpha + \beta u + \gamma u^2$, or sinh u. Suppose we consider the first. We have

$$L_x k(u) L_x u = c\rho L_t u$$

or

$$L_x k_0 L_x u + L_x k_1 u L_x u = c\rho L_t u$$

The procedure is straightforward. We have two linear terms $k_0 L_x L_x u$ and $c\rho L_t u$ and the nonlinear term $k_1 L_x(uL_x u)$. In the nonlinear term, we replace $uL_x u = uu_x$ by $\sum_{n=0}^{\infty} A_n$ where the A_n are generated for $Nu = uu_x$. We then solve for the two linear terms in turn using the inverse operators L_x^{-1} and L_t^{-1}. We define u_0 as usual and assume the decomposition $u = \sum_{n=0}^{\infty} u_n$. The components $u_1, u_2, \ldots$ now are determinable.

Solving for the linear term $L_x k_0 L_x u$ and substituting α for $c\rho$,

$$k_0 L_x u = u(0,t) + L_x^{-1}\alpha L_t u - L_x^{-1} L_x k_1 u L_x u$$

$$L_x u = k_0^{-1} u(0,t) + k_0^{-1} L_x^{-1}\alpha L_t u - k_0^{-1} L_x^{-1} L_x k_1 u L_x u$$

$$u = u(0,t) + L_{xx}^{-1}k_0^{-1}u(0,t) + L_{xx}^{-1}k_0^{-1}L_{xx}^{-1}\alpha L_t u$$

$$- L_{xx}^{-1}k_0^{-1}L_{xx}^{-1}L_{xx}k_1 u L_{xx}u$$

$$u = u(0,t) + L_{xx}^{-1}k_0^{-1}u(0,t) + k_0^{-1}\alpha L_{xx}^{-1}L_{xx}^{-1}L_t u$$

$$- L_{xx}^{-1}k_0^{-1}L_{xx}L_{xx}^{-1}k_1 u L_{xx}u$$

Now, solving for the remaining linear term

$$L_t u = \alpha^{-1}L_{xx}k_0 L_{xx}u + \alpha^{-1}L_{xx}k_1 L_{xx}u$$

$$u = u(x,0) + \alpha^{-1}k_0 L_t^{-1}L_{xx}L_{xx}u + \alpha^{-1}k_1 L_t^{-1}L_{xx}uL_{xx}u$$

Adding the two equations for u and dividing by two:

$$u_0 = (1/2)\{u(0,t) + u(x,0) + L_{xx}^{-1}k_0^{-1}u(0,t)\}$$

$$u_1 = (1/2)\{k_0^{-1}\alpha L_{xx}^{-1}L_{xx}^{-1}L_t u_0 - L_{xx}^{-1}k_0^{-1}L_{xx}^{-1}L_{xx}k_1 u_0 L_{xx}u_0$$

$$+ \alpha^{-1}k_0 L_t^{-1}L_{xx}L_{xx}u_0 + \alpha^{-1}k_1 L_t^{-1}L_{xx}u_0 L_{xx}u_0\}$$

$$u_2 = (1/2)\{k_0^{-1}\alpha L_{xx}^{-1}L_{xx}^{-1}L_t u_1 - L_{xx}^{-1}k_0^{-1}L_{xx}^{-1}L_{xx}k(u_1 L_{xx}u_0 + u_0 L_{xx}u_1)$$

$$+ \alpha^{-1}k_0 L_t^{-1}L_{xx}L_{xx}u_1 + \alpha^{-1}k_1 L_t^{-1}L_{xx}(u_0 L_{xx}u_1 + u_1 L_{xx}u_0)\}$$

so that components are determinable as necessary.

11) Non-uniform Conduction or Nonlinear Boundary Conditions: Suppose the heat conducting material is not uniform so that the coefficients in the equation are discontinuous at one or even several points. We can then divide the total interval $[0,\ell]$ at the points ε_i of discontinuity into subintervals. If the temperature and the heat flow are continuous at the points ε_i, we have

$$u(\varepsilon_i - 0,t) = u(\varepsilon_i + 0,t)$$

$$k(\varepsilon_i - 0)\partial u/\partial x(\varepsilon_i - 0,t) = k(\varepsilon_i - 0)\partial u/\partial x(\varepsilon_i + 0,t)$$

If heat is being radiated at the $x = 0$ cross section of the conducting material with the temperature $T(t)$, we have the nonlinear boundary condition:

$$k\ \partial u/\partial x(0,t) = \sigma[u^4(0,t) - T^4(0,t)]$$

Such nonlinear boundary conditions can also be handled by the decomposition method. The procedure is simple. We use the nonlinear boundary equation given just like a differential equation again using the approximation ϕ_n - for example, the three-term approximation ϕ_3 - as the solution of the boundary equation which results in evaluation of the constants of integration in ϕ_3. A large number of examples of this type have now been verified for accuracy.

The power and convenience of the decomposition method is shown by the following examples for comparisons with usual procedures.

Consider the homogeneous heat conduction equation $u_t = a^2u_{xx}$ with $u(x,0) = \phi(x)$ and $u(0,t) = u(\ell,t) = 0$.

Applying the well-known "separation of variables" technique assuming $\phi(x)$ is continuous, bounded, possesses piecewise continuous derivatives and satisfies $\phi(0) = \phi(\ell) = 0$. (Then $u(x,t)$ will be continuous for $t \geq 0$.) Let $u(x,t) = X(x)T(t)$. Then we have

$$(1/a^2)(T'/T) = (X'/X) = -\lambda = \text{constant}$$

which leads to the equations

$$X'' + \lambda X = 0 \qquad X(0) = X(\ell) = 0$$

and

$$\lambda_n = (\pi n/\ell)^2 \qquad \text{where } n = 1,2,3,\ldots$$

$$X_n(x) = \sin(\pi n x/\ell)^2$$

$$T_n(t) = C_n e^{-a\lambda_n t}$$

where the C_n *are constants. Thus*

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x)T_n(t) = \sum_{n=0}^{\infty} c_n e^{-a^2\lambda_n t}\sin(\pi n/\ell)x$$

satisfies the homogeneous boundary conditions. To satisfy the initial conditions

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} C_n \sin(\pi n x/\ell)$$

which is the Fourier sine series so that the c_n are Fourier coefficients which we can evaluate

$$c_n = \phi_n = 2/\ell \int_0^{\ell} \phi(\epsilon)\sin(\pi n\epsilon/\ell)d\epsilon$$

[It can be shown that the series converges and is appropriately (term-wise twice) differentiable with respect to x, and satisfies the differential

equation in $0 < x < \ell$, $t > 0$.] We will see below that solution by decomposition is very much simpler and that it results in a much more easily computed result. Before we proceed with decomposition we would like to comment that this solution can also be written as

$$u(x,t) = \int_0^{\ell} G(x,\varepsilon,t)\phi(\varepsilon)d\varepsilon$$

when the Green's function G is given by

$$G = (2/\ell) \sum_{n=1}^{\infty} \exp\{-(\pi n/\ell)^2 a^2 t\} \sin(\pi n x/\ell) \sin(\pi n \varepsilon/\ell)$$

which can be seen by looking at $u(x,t)$, substituting values of c_n. Since the series for c_n converges uniformly for $t > 0$ with respect to ε, the summation and the integration can be interchanged.

If instead of $u_t = a^2 u_{xx}$, we consider

$$u_t = a^2 u_{xx} + f(x,t)$$

i.e., a more general equation with a forcing function and assume given conditions as:

$$u(x,0) = u(0,t) = u(\ell,t) = 0$$

the solution becomes

$$u(x,t) = \int_0^t \int_0^{\ell} G(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau$$

$$G = (2/\ell) \sum_{n=1}^{\infty} e^{-(\pi n \ell)^2 a^2 (t-\tau)} \sin(\pi n x/\ell)\sin(\pi n \varepsilon/\ell).$$

We can, if we wish, apply decomposition to the separated equation, but we emphasize that there is no need for the separation anymore since

we can do the partial differential equation directly as demonstrated by our examples so far. The decomposition result is very much simpler and much easier to compute than the result above. If we add nonlinear or stochastic terms as well, then the decomposition method works as easily and yields a much superior result because no artificial simplifications are forced on us.

12) Self-shadowing Latticework Structures in Orbit: An orbiting latticework structure - a truss - undergoes time-dependent heating by the sun and the earth, as well as by power systems, heat-exchangers, and electronics on the space station. Meanwhile heat is being radiated by the truss itself.

The temperature distribution $T(\alpha,t)$ along a truss member is obtained by solving conservation-of-energy equations - ordinarily accomplished by finite-element methods.

Since the specific equation is not significant in this discussion and the heating history depends on the orbit, the shadowing by the rest of the structure, etc., let us first consider a simple case with an isothermal truss member. The truss-member has temperature $T(t)$ which must be found from

$$\partial T/\partial t + kT^4 = g$$

where k depends on density, emissivity, volume, emitted radiation surface area, and the Stefan-Boltzmann constant and g depends on incident radiation surface area and heating rate. The equation can be

rewritten in our form

$$L_t T = g - kT^4$$

Applying the inverse operator as usual, we have

$$T = T(0) + L_t^{-1} g - kL_t^{-1} T^4$$

and by decomposition

$$T_0 = T(0) + L_t^{-1} g$$

$$T_1 = -kL_t^{-1} A_0\{T^4\}$$

$$T_2 = -kL_t^{-1} A_1\{T^4\}$$

$$\vdots$$

$$T_{n+1} = -kL_t^{-1} A_n\{T^4\}$$

where $\sum_{n=0}^{\infty} T_n = T$. An n-term approximation will then be given by $\phi_n = \sum_{i=0}^{n-1} T_i$ or $T_0 + T_1 + \cdots + T_{n-1}$ which we have shown how to calculate above once we know the A_n. The A_n polynomials for the quadratic nonlinearity T^4 are available from the published literature* and are given by:

$$A_0 = T_0^4$$

$$A_1 = 4T_0^3 T_1$$

$$A_2 = 4T_0^3 T_2 + 6T_0^2 T_1^2$$

$$A_3 = 4T_0^3 T_3 + 4T_1^3 T_0 + 12T_0^2 T_1 T_2$$

*See reference [1,2,23].

$$A_4 = T_1^4 + 4T_0^3T_4 + 6T_0^2T_2^2 + 12T_0^2T_1T_3 + 12T_1^2T_0T_2$$

$$A_5 = 4T_0^3T_5 + 4T_1^3T_2 + 12T_0^2T_1T_4 + 12T_0^2T_2T_3 + 12T_1^2T_0T_3 + 12T_2^2T_0T_1$$

$$\vdots$$

Thus the system is completely calculable.

When we have a more complex situation with multiple truss-members, systems of coupled equations arise. These are solved with equal facility by decomposition.

Also, since heating will not be uniform, the isothermal approximation is obviously poor and we will have nonlinear partial differential equations such as

$$\partial T/\partial t - \alpha\partial^2 T/\partial x^2 + \beta T^4 = g$$

where α depends on thermal conductivity, cross-sectional area, density, specific heat, and volume and β depends on emissivity, emitted radiation surface area, and the Stefan-Boltzman constant.* In order to solve for T we write

$$L_t T - \alpha L_{xx} T + \beta T^4 = g$$

This we now solve by decomposition by first solving for the two linear operator terms; thus we get the two equations

$$L_t T = g + \alpha L_{xx} T - \beta T^4 \qquad (1)$$

$$\alpha L_{xx} T = -g + L_t T + \beta T^4 \qquad (2)$$

From equation (1) if α, β are constant

$$T = T(0) + L_t^{-1} g + \alpha L_t^{-1} L_{xx} T - \beta L_t^{-1} T^4 \qquad (3)$$

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*"Self-Shadowing Effects on Thermal Structural Response of Orbiting Stre-ses," *J. Mahoney and E. Thornton, J. Spacecraft, 24,* no. 4, pp. 342-348.

From equation (2)

$$T = A + Bx - \alpha^{-1}L_{xx}^{-1}g + \alpha^{-1}L_{xx}^{-1}L_t T + \alpha^{-1}\beta L_{xx}^{-1}T^4 \qquad (4)$$

With the decomposition of T into components to be found, i.e., $T = \sum_{n=0}^{\infty} T^n$ and representing the nonlinear term T^4 using the A_n polynomials for T^4, i.e., $T^4 = \sum_{n=0}^{\infty} A_n\{T^4\}$ we have

$$A_0 = T_0^4$$

$$A_1 = 4T_0^3T_1$$

$$A_2 = 4T_0^3T_2 + 6T_0^2T_1^2$$

$$A_3 = 4T_0^3T_3 + 4T_1^3T_0 + 12T_0^2T_1T_2$$

$$A_4 = T_1^4 + 4T_0^3T_4 + 6T_0^2T_2^2 + 12T_0^2T_1T_3 + 12T_1^2T_0T_2$$

$$A_5 = 4T_0^3T_5 + 4T_1^3T_2 + 12T_0^2T_1T_4 + 12T_0^2T_2T_3 + 12T_1^2T_0T_3 + 12T_2^2T_0T_1$$

$$\vdots$$

From equation (3)

$$T_0 = T(0) + L_t^{-1}g$$

$$T_{n+1} = \alpha L_t^{-1}L_{xx}T_n - \beta L_t^{-1}A_n$$

From equation (4)

$$T_0 = A + Bx$$

$$T_{n+1} = -\alpha^{-1}L_{xx}^{-1}g + \alpha^{-1}L_{xx}^{-1}L_tT_n + \alpha^{-1}\beta L_{xx}^{-1}A_n$$

Here A and B are determined by allowing T_0 or ϕ_1, the first or

one-term approximation to T, to satisfy given conditions $T(0,t)$ and $T(\ell,t)$ for the truss member temperature. Since T_0 is now known in both equations, the one-term approximate for T is ϕ_1, which is half the sum of the two ϕ_1's*

Now we can calculate the two u_1's and continue in the same manner until we have an n-term approximation ϕ_n, i.e., an approximation to the correct temperature T which becomes more accurate as we go to more terms until there is no further change.

It will now be clear that the decomposition method is not only applicable to generic space structure thermal problems but provides an extremely useful straightforward method as compared with ordinary numerical procedures.

Consider some examples carried out to a complete solution to demonstrate the validity of these procedures.

EXAMPLE 1:

Consider the application of decomposition to an equation $u_{xx} - u_{yy} = 0$ modelling a square plate with $0 \leq x \leq \pi/2$ and $0 \leq y \leq \pi/2$ with the given conditions:

$u(0,y) = 0$ $\qquad$ $u(\pi/2,y) = \sin y$

$u(x,0) = 0$ $\qquad$ $u(x,\pi/2) = \sin x$

*In some special cases, one equation may not contribute because the ϕ_1 term is zero. Then we simply work with the remaining equation and the adding and dividing is unnecessary. See example (4) on a following page.

Let $L_{xx} = \partial^2/\partial x^2$ and $L_{yy} = \partial^2/\partial y^2$ and write the above equation as $L_{xx}u = L_{yy}u$. As usual in the decomposition method, we solve for each linear operator term, $L_{xx}u$ and $L_{yy}u$, in turn and then apply the appropriate inverse to each.

$$L_{xx}^{-1}L_{xx}u = u - c_1k_1(y) - c_2k_2(y)x - L_{xx}^{-1}L_{yy}u$$

$$L_{yy}^{-1}L_{yy}u = u - c_3k_3(x) - c_4k_4(x)y - L_{yy}^{-1}L_{xx}u$$

or

$$u = c_1k_1(y) + c_2k_2(y)x + L_{xx}^{-1}L_{yy}u \tag{1}$$

$$u = c_3k_3(x) + c_4k_4(x)y + L_{yy}^{-1}L_{xx}u \tag{2}$$

Define $\phi_x = c_1k_1(y) + c_2k_2(y)x$ and $\phi_y = c_3k_3(x) + c_4k_4(x)y$ to rewrite (1) and (2) as

$$u = \phi_x + L_{xx}^{-1}L_{yy}u \tag{3}$$

$$u = \phi_y + L_{yy}^{-1}L_{xx}u \tag{4}$$

One-term approximants to the solution u are $u_0 = \phi_x$ in (3) and $u_0 = \phi_y$ in (4). Two-term approximants are $u_0 + u_1$ where $u_1 = L_{xx}^{-1}L_{yy}u_0$ in (3) and $L_{yy}^{-1}L_{xx}u_0$ in (4), etc. Thus $u_{n+1} = L_{xx}^{-1}L_{yy}u_n$ in (3) and $L_{yy}^{-1}L_{xx}u_n$ in (4) for $n \geq 0$.

For the x conditions $u(x,0) = 0$ and $u(x,\pi/2) = \sin x$ applied to the one-term approximant $u_0 = c_1k_1(y) + c_2k_2(y)x$, we have

$$c_1k_1(y) = 0$$

$$c_2k_2(y)\pi/2 = \sin y$$

or $c_2 = 2/\pi$ and $k_2(y) = \sin y$.

For the y conditions $u(x,0) = 0$ and $u(x,\pi/2) = \sin x$ applied to $u_0 = c_3k_3(x) + c_4k_4(x)y$, we get

$$c_3k_3(x) = 0$$

$$c_4k_4(x)\pi/2 = \sin x.$$

Thus $c_4 = 2/\pi$ and $k_4(x) = \sin x$.

If a one-term approximant were sufficient, the solution would be*

$$\phi_1 = (1/2)\{(2/\pi)x \sin y + (2/\pi)y \sin x\}$$

The next terms for (3) and (4) respectively are

$$u_1 = L_{xx}^{-1}L_{yy}u_0 = L_{xx}^{-1}L_{yy}[c_2x \sin y]$$

$$u_1 = L_{yy}^{-1}L_{xx}u_0 = L_{yy}^{-1}L_{xx}[c_4y \sin x]$$

We continue to obtain $u_2, u_3, \ldots$. Clearly, for any n,

$$u_n = (L_{xx}^{-1}L_{yy})^n u_0 = c_2(\sin y)(-1)^n x^{2n+1}/(2n+1)!$$

$$u_n = (L_{yy}^{-1}L_{xx})^n u_0 = c_4(\sin x)(-1)^n y^{2n+1}/(2n+1)!$$

Letting ϕ_m represent the m-term approximant, we have for the two cases:

$$\phi_m = c_2 \sin y \sum_{n=0}^{m-1} (-1)^n x^{2n+1}/(2n+1)! \qquad (5)$$

$$\phi_m = c_4 \sin x \sum_{n=0}^{m-1} (-1)^n y^{2n+1}/(2n+1)! \qquad (6)$$

We can now apply the conditions $\phi_m(\pi/2,y) = \sin y$ for (5); thus

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*An n-term approximant is $\phi_n = \sum_{i=0}^{n-1} u_i$

$$c_1 k_1(y) = 0$$

$$c_2 \sin y \sum_{n=0}^{m-1} (\pi/2)^{2n+1}/(2n+1)! = \sin y$$

$$c_2 = \frac{1}{\sum_{n=0}^{m-1} (-1)^n (\pi/2)^{2n+1}/(2n+1)!}$$

As m approaches infinite, we recognize the denominator as $\sin \pi/2$ so that c_2 approaches 1 and the sum in (5) approaches $\sin x$ in the limit.

Now applying the conditions $\phi_m(x,0) = 0$ and $\phi_m(x,\pi/2) = \sin x$, we have

$$c_3 k_3(x) = 0$$

$$c_4 \sin x \sum_{n=0}^{m-1} (-1)^n (\pi/2)^{2n+1}/(2n+1)! = \sin x$$

$$c_4 = \frac{1}{\sum_{n=0}^{m-1} (-1)^n (\pi/2)^{2n+1}/(2n+1)!}$$

Again as $m \to \infty$, $c_4 \to 1$ and the sum in (6) becomes $\sin y$. We can now write the exact solution

$$u = (1/2)\{\sin y \sin x + \sin x \sin y\}$$

or

$$u = \sin y \sin x.$$

So for this case, the series is summed into a convenient closed form solution.

EXAMPLE II:

If we modify our example to $u_{xx} - u_{yy} + f(u) = 0$ we let $f(u) = \sum_{n=0}^{\infty} A_n\{f(u)\}$, i.e., $f(u)$ is represented by the A_n polynomials generated specifically for $f(u)$. Then the equations (3) and (4) in the previous example become

$$u = \phi_x + L_{xx}^{-1}L_{yy} \sum_{n=0}^{\infty} u_n - L_{xx}^{-1} \sum_{n=0}^{\infty} A_n\{f(u)\}$$

and

$$u = \phi_y + L_{yy}^{-1}L_{xx} \sum_{n=0}^{\infty} u_n + L_{yy}^{-1} \sum_{n=0}^{\infty} A_n\{f(u)\}$$

and computation proceeds as before.

EXAMPLE III:

Now consider $u_{xx} = u_{tt}$ on $0 \leq x \leq \pi$ with $t \geq 0$ specifying

$u(x,0) = \sin x$ $\qquad$ $u(0,t) = 0$

$u_t(x,0) = 0$ $\qquad$ $u(\pi,t) = 0$

Decomposition yields

$$u = c_1k_1(t) + c_2k_2(t)x + L_{xx}^{-1}L_{tt} \sum_{n=0}^{\infty} u_n \qquad (1)$$

$$u = c_3k_3(x) + c_4k_4(x)t + L_{tt}^{-1}L_{xx} \sum_{n=0}^{\infty} u_n \qquad (2)$$

In (1) the $u_0 = c_1k_1(t) + c_2k_2(t)x$ term vanishes when the boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$ are applied. Hence in this

case (1) does not contribute and we need consider only (2). Here $u_0 = c_3k_3(x) + c_4k_4(x)t$ and applying the conditions $u(x,0) = \sin x$ and $u_t(x,0) = 0$, we have

$$u_0 = \sin x$$

$$u_1 = L_{tt}^{-1}L_{xx}u_0 = (-t^2/2) \sin x$$

$$u_2 = L_{tt}^{-1}L_{xx}u_1 = (t^4/4!) \sin x$$

$$\vdots$$

$$u = \sin x \cos t$$

Here, we recognized the series for $\cos t$ but ordinarily use the series and see, most effectively, in numerical solutions that the solutions stabilize as n increases, i.e., the approximant $\phi_n = \sum_{i=0}^{n-1} u_i$ converges to the correct solution $u = \sum_{n=0}^{\infty} u_n$.

EXAMPLE IV:

We point out an example where one linear operator term does not contribute. Consider the example $u_{xx} = u_t$ with given conditions

$$u(x,0) = \sin \pi x \quad \text{for} \quad 0 < x < 1$$

$$u(0,t) = u(1,t) = 0 \quad \text{for} \quad t > 0$$

We write

$$L_{xx}u = L_t u \tag{1}$$

$$L_t u = L_{xx}u \tag{2}$$

Applying the L_{xx}^{-1} to (1) we have

$$u = u_0 + L_{xx}^{-1}L_t u$$

$$u_0 = A(t) + B(t)x = 0$$

because of the conditions $u(0,t) = u(1,t) = 0$; hence this equation doesn't contribute and under the rules of the decomposition method is not considered in determining u_0.

From (2) applying L_t^{-1}, we have

$$u = u_0 + L_t^{-1}L_{xx}u$$

$$u_0 = u(x,0) = \sin \pi x$$

$$u_1 = L_t^{-1}L_{xx}u_0 = L_t^{-1}L_{xx} \sin \pi x$$

$$= -\pi^2 t \sin \pi x$$

$$u_2 = L_t^{-1}L_{xx}u_1$$

$$\vdots$$

Summarizing the series, we determine

$$u = e^{-\pi^2 t}\sin \pi x$$

is the exact solution since

$$u = u_0 + u_1 + \cdots$$

$$= \sin \pi x - \pi^2 t \sin \pi x + \cdots$$

$$= [1 - \pi^2 t + \cdots] \sin \pi x$$

where the bracketed quantity is the series for $e^{-\pi^2 t}$

If we changed the x interval to $(0,\pi/2)$ and specified $u(x,0) = \sin x$, $u(0,t) = 0$, and $u(\pi/2,t) = e^{-t}$, we get a case which is less convenient computationally because of the integration "constants." We no longer have $u_0 = 0$ for the $L_{xx}u$ equation and must consider both equations in determining the initial term u_0.

Now $u_0 = (2/\pi)xe^{-t}$. When we determine u_1, we get $-(2/\pi)(x^3/3!)e^{-t} + c_1(t)x + c_2(t)$ because of the two-fold integration represented by L_{xx}^{-1}.

When $x = 0$, $u = 0$; hence $c_2 = 0$. When $x = \pi/2$, $u = e^{-t}$; hence $(2/\pi)(\pi/2)^3(1/3!)e^{-t} + c_1(\pi/2) = e^{-t}$ determines c_1, etc., and we see this case will become numerically more complex. The solution $u = e^{-t}\sin x$ is still valid but not as simple to obtain.

We might also note that if $u(x,0)$ is specified, then u_{xx} is specified at $t = 0$ and therefore $u_t(x,0)$ is specified through the equation so we cannot independently specify $u_t(x,0)$. Thus the conditions cannot be assigned arbitrarily and must of course be physically appropriate. We can specify here the temperature u or the heat flow (derivative u_x) at both ends or, say, temperature at $x = 0$ and heat flow at $x = 1$ (rod with uninsulated end).

13) Transient Heat Transfer: K. N. Shukla and L. Mani (*AIAAJ*, October 1984) considered thermal constriction resistance with arbitrary heating in a convectively cooled plate. Precise heat transfer calculations are, of course, important to achieve reliability in space structures. The two-dimensional model considered in the above reference can be written in

our notation as: $L_{xx}\theta + L_{yy}\theta = kL_t\theta$ where $L_{xx} = \partial^2/\partial x^2$, $L_{yy} = \partial^2/\partial y^2$. We write immediately

$$L_{xx}\theta = kL_t\theta - L_{yy}\theta$$

$$L_{yy}\theta = kL_t\theta - L_{xx}\theta$$

$$L_t\theta = k^{-1}L_{xx}\theta + k^{-1}L_{yy}\theta$$

With our inversions

$$\theta = A + Bx + kL_{xx}^{-1}L_t\theta - L_{xx}^{-1}L_{yy}\theta$$

$$\theta = C + Dy + kL_{yy}^{-1}L_t\theta - L_{yy}^{-1}L_{xx}\theta$$

$$\theta = \theta(t = 0) + k^{-1}L_t^{-1}L_{xx}\theta + k^{-1}L_t^{-1}L_{yy}\theta$$

where the functions A, B, C, D are determined from specified boundary conditions (and the initial condition appears in the last equation for θ). Once these boundary conditions are specified, the components of θ are determined by decomposition to some n-term approximation. These conditions are:

$\theta_x = 0$ at $x = 0$ and $x = \ell$

$\theta(t = 0)$ is a specified function $f(x,y)$

$\theta_y(y = 0)$ is a specified function $g(x)$

$|\theta_y + B\theta|_\alpha = 0$

where α is a constant.

Decomposition calculations are simpler than by other methods, and, most important, solutions especially in nonlinear or stochastic cases become

more realistic. Of course, modelling is necessary before we can apply any method and we must begin with a model. Now that we know decomposition works, we need not begin with simplistic models.

14) Transient heat conduction: An equation describing transient heat conduction is a region R is

$$\nabla^2 u + k^{-1} f(x,y,z,t) = \alpha^{-1} \partial u/\partial t$$

u is temperature, k is thermal conductivity, and α is thermal diffusivity. k and α are constants. Six boundary conditions and an initial condition are required. (Such a problem is described by Beck in *AIAA Journal, 24,* no. 2, Feb. 1984, pp. 327-333 with the use of Butkovsky's Green's functions*.) In our form, we write

$$[L_{xx} + L_{yy} + L_{zz}]u + k^{-1} f = \alpha^{-1} L_t u$$

We solve solve for the four linear operator terms, apply the appropriate inversions, let $u = \sum_{n=0}^{\infty} u_n$ and $f = \sum_{n=0}^{\infty} A_n$ where the A_n are generated for the function f.

$$L_{xx}u = \alpha^{-1} L_t u - L_{yy}u - L_{zz}u - k^{-1} f$$

$$L_{yy}u = \alpha^{-1} L_t u - L_{zz}u - L_{xx}u - k^{-1} f$$

$$L_{zz}u = \alpha^{-1} L_t u - L_{xx}u - L_{yy}u - k^{-1} f$$

$$L_t u = \alpha L_{xx}u + \alpha L_{yy}u + \alpha L_{zz}u + k^{-1} f$$

*A. G. Butkovsky, Green's Functions and Transfer Functions Handbook, Wiley, 1982.

or

$$u = \phi_{xx} + \alpha^{-1}L_{xx}^{-1}L_t u - L_{xx}^{-1}L_{yy}u - L_{xx}^{-1}L_{zz}u - k^{-1}L_{xx}^{-1}f$$

$$u = \phi_{yy} + \alpha^{-1}L_{yy}^{-1}L_t u - L_{yy}^{-1}L_{zz}u - L_{yy}^{-1}L_{xx}u - k^{-1}L_{yy}^{-1}f$$

$$u = \phi_{zz} + \alpha^{-1}L_{zz}^{-1}L_t u - L_{zz}^{-1}L_{xx}u - L_{zz}^{-1}L_{yy}u - k^{-1}L_{zz}^{-1}f$$

$$u = \phi_t + \alpha L_t^{-1}L_{xx}u + \alpha L_t^{-1}L_{yy}u + \alpha L_t^{-1}L_{zz}u + k^{-1}L_t^{-1}f$$

ϕ_t satisfies the initial condition $u(x,y,z,0)$ and the ϕ_{xx}, ϕ_{yy}, ϕ_{zz} each satisfy two boundary conditions. The ϕ_{xx}, ϕ_{yy}, ϕ_{zz}, ϕ_t are the u_0 terms of the decomposition for each equation. Then since $u = \sum_{n=0}^{\infty} u_n$ and $f = \sum_{n=0}^{\infty} A_n$ the following components are determined. For example

$$u_1 = \alpha^{-1}L_{xx}^{-1}L_t u_t - L_{xx}^{-1}L_{yy}u_0 - L_{xx}^{-1}L_{zz}u_0 - k^{-1}L_{xx}^{-1}A_0$$

$$u_1 = \alpha^{-1}L_{yy}^{-1}L_t u_0 - L_{yy}^{-1}L_{zz}u_0 - L_{yy}^{-1}L_{xx}u_0 - k^{-1}L_{yy}^{-1}A_0$$

$$u_1 = \alpha^{-1}L_{zz}^{-1}L_t u_0 - L_{zz}^{-1}L_{xx}u_0 - L_{zz}^{-1}L_{yy}u_0 - k^{-1}L_{zz}^{-1}A_0$$

$$u_1 = \alpha L_t^{-1}L_{xx}u_0 + \alpha L_t^{-1}L_{yy}u_0 + \alpha L_t^{-1}L_{zz}u_0 + k^{-1}L_t^{-1}A_0$$

etc., for u_2, u_3, ... The solution u is obtained by adding the four equations for u and dividing by 4, or, as an approximation, by using the four approximate solutions ϕ_n.

NONLINEAR OSCILLATIONS

Introduction: Practically all the problems of mechanics are nonlinear at the outset. Nonlinear oscillating systems are generally analyzed by approximation methods which involve some sort of *linearization*. These replace an actual nonlinear system with a so-called "equivalent" linear system and employ averagings which are not generally valid.

While the linearizations commonly used are adequate in some cases, they may be grossly inadequate in others. Thus, if we want to know how a physical system behaves, it is essential to retain the nonlinearity, not just solve a convenient mathematized model.

Using the decomposition method, restrictive assumptions on nonlinear and stochastic behavior for the sake of a tractable model, become unnecessary; correct solutions are obtained in the strongly nonlinear case and in the case of stochastic (large fluctuation) behavior, as well as in the cases where perturbation would be applicable or in the linear and/or deterministic limits.

We are concerned in this section with the study of vibrations or oscillatory motion and the associated forces. Vibrations can occur in any mechanical system having mass and elasticity. Consequently, they can occur in structures and machines of all kinds. In large space structures contatining men or machinery, such vibrations will result in difficult and crucial control problems and also lifetime or duration considerations since vibrations can lead to eventual failure.

Oscillations can be regular and periodic, or they can be random. Randomness leads to stochastic differential equations. In deterministic

systems - the special case where randomness vanishes - the equations modeling the system provide instantaneous values for any time. When random functions are involved, the instantaneous values are unpredictable, and it is necessary to resort to a statistical description. Such random functions of time, or stochastic processes, occur in problems, for example, such as pressure gusts encountered by aircraft, jet engine noise, or ground motion in earthquakes.

Oscillatory motion is modelled by equations of the general form:

$$y'' + f(y,y',t) = x(t)$$

$$Fy = x(t)$$

Stochastic processes may be involved in coefficients, input terms, or initial boundary conditions, so $x(t)$ can be assumed to be generally stochastic. Input conditions - possibly stochastic - are given (statistically described if stochastic). Therefore F will be a nonlinear stochastic operator with (possibly) stochastic coefficients. More conveniently, we write $Fy = Ly + Ny$ where Ly is a linear (stochastic) term, and Ny is a nonlinear (stochastic) term, or

$$Ly + Ny = x.$$

Now, for the linear operator L, define $L = \mathrm{L} + \mathrm{R} + R$ where: L is a linear invertible deterministic operator, so that L^{-1} exists; R is the remainder of the linear deterministic operator, vanishing, of course, if we can easily invert the entire linear deterministic operator, and R is a (linear) stochastic operator.

Similarly for the nonlinear operator N, define $Ny = Ny + My$ since there may be both deterministic nonlinear terms Ny and stochastic (nonlinear) terms My.

Thus in general we rewrite our general oscillator equation as

$$Ly + Ry + Ry + Ny + My = x.$$

Note that in a particular problem any number of terms from one to four may vanish. Still more generally, Ny may actually be a function of $y,y',\ldots$ but we will again write Ny for simplicity. Calculation of the A_n for such a case becomes more difficult and will not be discussed here. Our purpose now is not discussion of the entire mathematical theory but demonstration of applicability to space structures.

Convergence of the decomposition series can reasonably be expected to be fastest when we invert the entire linear deterministic operator. Computation of the integrals will, however, be more difficult should this be the case, since the Green's functions will not then be simple. Thus, for ease of computation we will let L denote the highest order differential operator or d^2/dt^2 in the above oscillator equation.

Let us review the significance of the mathematical expressions we will deal with in order to make the work as clear as possible. In an oscillator, we have generally an external force or driving term $x(t)$, a restoring force $f(y)$ dependent on the displacement y, and a damping force, since energy is always dissipated in friction or resistance to motion. Usually this is dependent on velocity, and we will write it as $g(y')$.

If we have a free oscillating mass m on a spring with no damping, we can write

$$my'' + ky = 0$$

if the spring obeys Hooke's Law, i.e., assuming displacement proportional to force. Of course, no spring really behaves this way. Often the force needed for a given compression is not the same as for an extension of the same amount. Such asymmetry is represented by a quadratic force, or force proportional to y^2 rather than y. We may have a symmetric behavior with proportionality to y^3. In this case, while the solution is not the harmonic solution which one gets for the model equation $my'' + ky = 0$, it is still a periodic solution. The damping force $g(y')$ may be cy' where c is constant, or it may be more complicated such as a nonlinear function of y or its derivatives, e.g., $g(y',y'^2)$ so that it depends on v^2 as well as v. By usual methods, analytic solutions then become impossible, i.e., we can not deal analytically with such a case unless we *resort to linearization.*

If we write $-f(y)$ for the restoring force, $-h(y')$ for the damping force, and represent the driving force with g; the resulting equation will be

$$y'' + f(y) + h(y') = g$$

Suppose the restoring force is represented by an odd function such that $f(y) = -f(-y)$. We have this in most applications; it means simply that if we reverse the displacement, then the restoring force reverses its direction. A pendulum, for example, behaves this way. We might take the first two terms of the power series for $f(y)$ and write $f(y) = \alpha y + \beta y^3$. Then we have

$$y'' + \alpha y + \beta y^3 = g$$

If we have damping also, we have

$$y'' + cy' + \alpha y + \beta y^3 = g$$

assuming the damping force is $-cy'$. (This is *Duffing's equation.*)

We will begin - after a simple example of a pendulum (harmonic motion) - with the well-known anharmonic oscillator then go on to consider more general oscillators such as the *Duffing oscillator* and the *Van der Pol oscillator.* The Duffing oscillator in a random force field is modeled by $y'' + \alpha y' + \beta y + \gamma y^3 = x(t)$. It can be analyzed without limiting the force $x(t)$ to a white noise or restricting α, β, γ to be deterministic. The same applies to the Van der Pol oscillator modeled by $y'' + \varepsilon y^2 y' - \varepsilon y' + y = x(t)$. These equations are in our standard from $Fy = x(t)$ which can be solved by the decomposition method. If the equation is linear and deterministic, we have $Fy = Ly = x$. An equation that is deterministic but nonlinear can be rewritten as $Fy = Ly + Ny = x$. A linear stochastic equation is $Ly = x$, etc. These cases are discussed in the earlier work. Consider the pendulum problem as an example, then we proceed with the anharmonic oscillator.

A Simple Nonlinear Oscillator Example:

Consider the simple vertical pendulum consisting of a mass m at the end of a rod length ℓ moved through an angle θ. We obtain immediately

$$d^2\theta/dt^2 + k^2 \sin\theta = 0$$

where $k^2 = g/\ell$. Let $L = d^2/dt^2$ and $N(\theta) = k^2 \sin\theta$ to obtain our usual standard form $L\theta + N\theta = 0$ for a homogeneous deterministic differential equation.

Small-amplitude case: The usual treatment is to simplify this anharmonic motion by assuming $\sin\theta \approx \theta$. We can then write $d^2\theta/dt^2 + k^2\theta = 0$, the well-known harmonic oscillator problem. L is defined as before, but now there is no nonlinear term, and the remainder of the linear term is $R\theta$, i.e., the entire linear operator is decomposed into $L + R$ where R is a linear deterministic "remainder" operator. Define L^{-1} as the double definite integration from 0 to t. If we have a forcing term as well, we would have $L\theta + R\theta = x(t)$. The solution is found by writing

$$L\theta = x - R\theta$$

$$L^{-1}L\theta = \theta - \theta(0) - t\theta'(0) = L^{-1}x - L^{-1}R\theta$$

$$\theta(t) = \theta_0 - L^{-1}R\theta$$

with $\theta_0 = \theta(0) + t\theta'(0) + L^{-1}x(t)$. Now substituting $\theta = \sum_{n=0}^{\infty} \theta_n(t)$, we find

$$\theta_{n\geq 1} = -L^{-1}R\theta_n$$

so that all components can now be determined.

Let us assume initial conditions are given as $\theta(0) = \gamma$ and $\theta'(0) = 0$. Then, since $x = 0$

$$\theta_0 = \gamma$$

$$\theta_1 = -L^{-1}R\theta_0 = -L^{-1}k^2\gamma = -\gamma k^2 t^2/2!$$

$$\theta_2 = -L^{-1}R\theta_1 = -L^{-1}k^2(-\gamma k^2 t^2/2!) = \gamma k^4 t^4/4!$$

$$\theta_3 = -\gamma k^6 t^6/6!$$

We have

$$\theta_n = \gamma(-1)^{n-1}(kt)^{2n-2}/(2n-2)!$$

$$\theta = \gamma \sum_{n=1}^{\infty} (-1)^{n-1}(kt)^{2n-2}/(2n-2)!$$

Thus $\theta = \gamma \cos kt$ with $k = (g)^{1/2}$.

If we assume initial conditions $\theta(0) = 0$, $\theta'(0) = \omega$, we obtain

$$\theta_0 = \omega t = (\omega/k)kt$$

$$\theta_1 = -(\omega/k)k^3t^3/3!$$

$$\theta_2 = (\omega/k)k^5t^5/5!$$

$$\vdots$$

$$\theta_n = (\omega/k)(-1)^{n-1}(kt)^{2n-1}/(2n-1)!$$

and finally,

$$\theta(t) = (\omega/k)\{kt - (kt)^3/3! + (kt)^5/5! - \ldots\}$$

$$= (\omega/k) \sum_{n=1}^{\infty} (-1)^{n-1}(kt)^{2n-1}/(2n-1)!$$

$$\theta(t) = (\omega/k) \sin kt$$

The case $\theta(0) = \gamma$, and $\theta'(0) = \omega$ yields

$$\theta_0 = \gamma + (\omega/k)kt$$

$$\theta_1 = -L^{-1}R[\gamma + (\omega/k)kt]$$

$$\vdots$$

$$\theta_n = \gamma(-1)^{n-1} (kt)^{2n-2}/ (2n-2)!$$

$$+ (\omega/k)(-1)^{n-1} (kt)^{2n-1}/(2n-1)!$$

AD-A190 143 A NEW APPROACH TO THE ANALYSIS AND CONTROL OF LARGE SPACE STRUCTURES PHASE 1(U) GENERAL ANALYTIC CORP ATHENS GA G ADOMIAN 12 MAR 88 GAC-881 AFOSR-TR-88-0702 2/2

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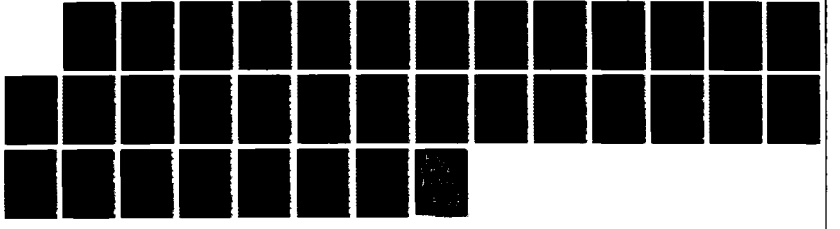

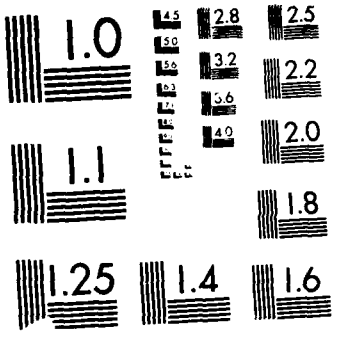

UTION TEST CHART

or

$$\theta(t) = \gamma \cos kt + (\omega/k) \sin kt$$

the well-known general solution of the harmonic oscillator.

Large-amplitude case: We can generalize to the anharmonic oscillator describing the pendulum problem for large amplitude motion, the only difference being that since we now have a nonlinearity, it must be expressed in terms of the A_n polynomials.

Consider the equation now for large-amplitude motion in the pendulum

$$d^2\theta/dt^2 + k^2 \sin \theta = 0$$

with $k^2 = g/\ell$. Let $L\theta = d^2\theta/dt^2$ and $N(\theta) = k^2 \sin \theta$. We let the nonlinear term $N\theta$ be represented by the sum of the A_n polynomials generated fro the nonlinear term as usual in the decomposition method. Thus $N\theta = \sum_{n=0}^{\infty} A_n$ where the A_n are given by:

$$A_0 = \sin \theta_0$$

$$A_1 = \theta_1 \cos \theta_0$$

$$A_2 = -(\theta_1^2/2) \sin \theta_0 + \theta_2 \cos \theta_0$$

$$A_3 = -(\theta_1^3/6) \cos \theta_0 - \theta_1\theta_2 \sin \theta_0 + \theta_3 \cos \theta_0$$

$$\vdots$$

This equation is a nonlinear deterministic homogeneous second-order differential equation. We must assume appropriate initial conditions. We will choose here $\theta(0) = \gamma$ = constant and $\theta'(0) = 0$. We can do it just as well for other combinations of initial conditions, of course, as will be obvious. We now have

$$L\theta = -N(\theta)$$

Operating with the inverse of L, a two-fold integration, we have

$$\theta = \theta_0 - L^{-1}N(\theta)$$

where

$$\theta_0 = \theta(C) + t\theta'(0)$$

which in our case is just γ. The $N(\theta)$ term - we will henceforth drop the parentheses - is replaced by the $\sum_{n=0}^{\infty} A_n$. We now have

$$\theta = \gamma - L^{-1} \sum_{n=0}^{\infty} A_n(k^2 \sin \theta)$$

We prefer to write

$$\theta = \gamma - L^{-1}k^2 \sum_{n=0}^{\infty} A_n(\sin \theta)$$

i.e., we can let $N\theta = \sin \theta$ rather than $k^2 \sin \theta$ since k^2 is a constant. The corresponding $A_n(\sin \theta)$, or simply A_n, are [1]:

$$A_0 = \sin \theta_0$$

$$A_1 = \theta_1 \cos \theta_0$$

$$A_2 = -(\theta_1^2/2) \sin \theta_0 + \theta_2 \cos \theta_0$$

$$A_3 = -(\theta_1^3/6) \cos \theta_0 - \theta_1\theta_2 \sin \theta_0 + \theta_3 \cos \theta_0$$

$$\vdots$$

Hence,

$$\theta_0 = \gamma$$

$$\theta_1 = -L^{-1}A_0$$

$$\theta_2 = -L^{-1}A_1$$

etc., to yield

$$\theta_1 = -\int_0^t dt \int_0^t dt\, k^2 \sin\theta_0 = -\int_0^t dt \int_0^t dt\, k^2 \sin\gamma$$

$$= -(\sin\gamma)(k^2t^2/2!)$$

$$\theta_2 = -\int_0^t dt \int_0^t dt\, k^2(\theta_1 \cos\theta_0)$$

$$= -\int_0^t dt \int_0^t dt\, k^2[-(k^2t^2/2!)\sin\gamma]\cos\gamma$$

$$= (k^4t^4/4!)\sin\gamma\cos\gamma$$

$$\theta_3 = -\int_0^t dt \int_0^t dt\, k^2[-(\theta_1^2/2)\sin\theta_0 + \theta_2\cos\theta_0]$$

$$= -(k^6t^6/6!)\cdot[\sin\gamma\cos^2\gamma - 3\sin^3\gamma]$$

etc. Thus,

$$\theta(t) = \gamma - [(kt)^2/2!]\sin\gamma + [(kt)^4/4!]\sin\gamma\cos\gamma$$

$$- [(kt)^6/6!)][\sin\gamma\cos^2\gamma - 3\sin^3\gamma]$$

$$+ [(kt)^8/8!][-33\sin^3\gamma\cos\gamma + \sin\gamma\cos^3\gamma]$$

$$- \ldots$$

i.e., we have the solution for large-amplitude motion.

As a check we can let γ be sufficiently small so that small amplitude motion is being considered. Then

$$\theta(t) = \gamma[1 - (kt)^2/2!) + (kt)^4/4! - \cdots]$$

which is, of course, the result for the linear harmonic oscillator with the given initial condition.

15) The Duffing and Van der Pol Oscillators: In the preceding section the decomposition method was applied to harmonic and anharmonic oscillators. We now apply it to the Duffing oscillator and the Van der Pol oscillator. It requires no linearization or "smallness" assumptions. The treatment can also include randomness in coefficients or inputs without customary restrictions to special processes or perturbation theory.

When we consider, for example, the equation for a simple pendulum, we approximate $\sin x$ by x to obtain the harmonic oscillator equation. Suppose we go a step further and write $\sin x = x - x^3/3!$, i.e., use the first two terms of the series for $\sin x$ assuming small x. We then get the equation $x'' + \omega^2 x + \varepsilon x^3 = 0$, which is the Duffing equation with ε as a "small" parameter. This is an example of a perturbation method; one would seek a solution in the form $x(t) = x_0(t) + \varepsilon x_1(t) + \ldots$. Using decomposition, we are not restricted to assuming small parameters.

Statistical Linearization: In dealing with stochastic oscillators, we depart again from usual procedures which require some sort of approximation

in order to determine the second-order response statistics. A common procedure in this connection is *statistical linearization*. This procedure simply replaces the original nonlinear equation with a so-called "equivalent" linear system. Thus, if we write an oscillator equation in the form:

$$x'' + \alpha x' + \omega_0 x + \beta f(x) = F(t)$$

where $x(t)$ is a displacement, α is a damping constant, ω_0 is a linear frequency, $\beta f(x)$ is a nonlinear restoring force, and $F(t)$ is a stationary process, the process of statistical linearization substitutes

$$x'' = \alpha x' + \gamma^2 x = F(t)$$

where γ^2 is determined in such a way that the mean square error due to the replacement is minimized, and the mean displacement is the same for both systems. It is customary to assume $F(t)$ is Gaussian and delta-correlated with zero-mean, or, $\langle F(t) \rangle = 0$ and $\langle F(t)F(t') \rangle = D\delta(t-t')$. This latter assumption is, of course, made for mathematical, not physical, reasons and is physically unrealistic. We propose none of these restrictions and will solve the actual nonlinear equation.

The Decomposition Method: The *Duffing Oscillator* is described by the equation:

$$y'' + \alpha y' + \beta y + \gamma y^3 = x(t)$$

In our standard $Fy = x(t)$ form.

The *Van der Pol equation* is generally given as:

$$y'' + \varepsilon y^2 y' - \varepsilon y' + y = x(t)$$

or by

$$y'' + \varepsilon y'(y^2 - 1) + y = x(t)$$

which we can write in the form

$$y'' + \alpha y' + \beta y + \gamma(d/dt)y^3 = x(t)$$

since $y^2y' = (d/dt)(y^3/3)$. Thus $\alpha = -\varepsilon$, $\beta = 1$, $\gamma = \varepsilon/3$ relates the last equation to the two previously given forms. We now have our standard form $Fy = Ly + Ny = x$, or $Fy = Ly + Ny = x$ (if no stochasticity is involved). Here L is assumed to be a linear and invertible operator and N is a nonlinear operator. We will consider the equations to be deterministic here.

The linear operator L in both the Duffing and Van der Pol oscillator equations is given by $d^2/dt^2 + \alpha d/dt + \beta$. The nonlinear term Ny is a simple cubic nonlinearity γy^3 in the case of the Duffing oscillator, and $\gamma(d/dt)y^3$ in the case of the Van der Pol oscillator. These terms will, as usual, be expanded in our A_n polynomials generated for the specific nonlinearity.

The treatment of the linear operator offers some alternatives. We can use the entire linear operator as L which enhances speed of convergence, but the inverse and consequent integrations become more difficult. We can also use part of the above operator which could be $L = d^2/dt^2$, $L = d^2/dt^2 + \alpha d/dt$, or $L = d^2/dt^2 + \beta$. We prefer in most cases to use $L = d^2/dt^2$, i.e., the highest order differential operator.

We expect this to give the slowest convergence but much easier integrations and less actual computation time.

The remainder of the linear operator will be called R, the "remainder" operator. If $L = d^2/dt^2$, $R = \alpha d/dt + \beta$. (If we also consider stochasticity, we will use a script letter $\mathcal{R}$ for a random part of the operator and may have $L + R + \mathcal{R}$.)

The choice made here (that $L = d^2/dt^2$) yields the simplest Green's function for computation. In this case, L^{-1} is the two-fold definite integral from 0 to t. Generally, this choice of the highest ordered derivative for L is the most desirable because the integrations are the simplest. If we invert the entire linear operator, convergence is expected to be much faster. It is interesting to examine a compromise here which can be used to advantage on occasion.

If we choose $L = d^2/dt^2 + \beta$, $R = \alpha d/dt$, we gain something in convergence rate over the previous case and expect to lose something in ease of computability. The interesting aspect that suggests the compromise is that we see we will get sine and cosine functions for solutions of the homogeneous equation. For the Duffing equation we now have

$$Ly = x - Ry - \gamma y^3$$

$$y = c_1\phi_1(t) + c_2\phi_2(t) + L^{-1}x - L^{-1}Ry - \gamma L^{-1}y^3$$

where ϕ_1, ϕ_2 satisfy $L\phi = 0$ or $d^2\phi/dt^2 + \beta\phi = 0$. Consequently,

$$\phi_1(t) = \cos\sqrt{\beta}t$$

$$\phi_2(t) = (1/\sqrt{\beta}) \sin \sqrt{\beta}t$$

Now

$$\sum_{n=0}^{\infty} y_n = c_1\phi_1(t) + c_2\phi_2(t) + L^{-1}x - L^{-1}Ry - \gamma L^{-1} \sum_{n=0}^{\infty} A_n$$

or

$$\sum_{n=0}^{\infty} y_n = c_1 \cos \sqrt{\beta}t + (c_2/\sqrt{\beta}) \sin \sqrt{\beta}t + L^{-1}x$$

$$- \alpha L^{-1}(d/dt) \sum_{n=0}^{\infty} y_n - \gamma L^{-1} \sum_{n=0}^{\infty} A_n$$

where the A_n are the appropriate polynomials for $Ny = y^3$. These are given by:

$$A_0 = y_0^3$$

$$A_1 = 3y_0^2y_1$$

$$A_2 = 3y_0y_1^2 + 3y_0^2y_2$$

$$A_3 = y_1^3 + 6y_0y_1y_2 + 3y_0^2y_3$$

$$A_4 = 3y_1^2y_2 + 3y_0y_2^2 + 6y_0y_1y_3 + 3y_0^2y_4$$

$$A_5 = 3y_1y_2^2 + 3y_1^2y_3 + 6y_0y_2y_3 + 6y_0y_1y_4 + 3y_0^2y_5$$

$$A_6 = y_2^3 + 6y_1y_2y_3 + 3y_1^2y_4 + 3y_0y_3^2 + 6y_0y_2y_4 + 6y_0y_1y_5 + 3y_0^2y_6$$

$$\vdots$$

Since $L = d^2/dt^2 + \beta$ now, L^{-1} is no longer the simple two-fold integral, and we must determine the Green's function for this L. The G will satisfy the equation $LG(t,\tau) = \delta(t - \tau)$ or

$$d^2G(t,\tau)/dt^2 + \beta G(t,\tau) = \delta(t - \tau)$$

G, of course, is determinable in a number of ways. We will again use the decomposition method itself and so we prepare by writing

$$d^2G/dt^2 = \delta(t - \tau) - \beta G$$

Again we have a simple second-order operator to invert. Hence

$$G(t,\tau) = G(0,\tau) + tG_t(0,\tau) + L^{-1}\delta(t - \tau) - \beta L^{-1} \sum_{n=0}^{\infty} G_n$$

Thus

$$\sum_{n=0}^{\infty} G_n = G(0,\tau) + LG_t(0,\tau) + L^{-1}\delta(t - \tau) - \beta L^{-1} \sum_{n=0}^{\infty} G_n$$

$$= G_0 - \beta L^{-1} \sum_{n=0}^{\infty} G_n$$

where

$$G_0 = G(0,\tau) + G_t(0,\tau) + tH(t - \tau)$$

and

$$G_1 - -\beta L^{-1}G_0 = -\beta\{G(0,\tau)t^2/2! + G_t(0,\tau)t^3/3!$$

$$+ (t^3/3!)H(t - \tau)\}$$

$$= -\beta G(0,\tau)[t^2/2! + t^3/3!] - \beta(t^3/3!)H(t - \tau)$$

etc., for G_2, G_3, ... An appropriate n-term approximation can now be used in the L^{-1} integrations.

The example $my'' + \omega^2 y + \alpha y^3 = 0$ occurs in the theory of nonlinear vibrating mechanical systems and in some nonlinear electrical systems which is the Duffing case above (with no damping). Suppose we have the specified conditions $y = a$ at $t = 0$ and $y' = 0$ at $t = 0$. Write $L = d^2/dt^2$ and

$$Ly = -(\omega^2/m)y - (\alpha/m)y^3$$

$$y = y(0) - L^{-1}(\omega^2/m) \sum_{n=0}^{\infty} y_n - (\alpha/m) \sum_{n=0}^{\infty} A_n$$

where

$$y_0 = a$$

$$y_1 = -(\omega^2/m)L^{-1}y_0 - (\alpha/m)A_0$$

$$= -a\omega^2t^2/2m - \alpha a^3/m$$

$$\vdots$$

so that $y = a[1 - \omega^2/t^2/2m - \alpha a^2/m \ldots]$.

Solution of the General Case of Duffing's Equation: For the Duffing equation with $L = d^2/dt^2$ we have $y = \sum_{n=0}^{\infty} y_n$ where

$$y_0 = y(0) + ty'(0) + L^{-1}x(t)$$

$$y_1 = -L^{-1}\alpha(d/dt)y_0 - L^{-1}\beta y_0 - L^{-1}\gamma A_0$$

$$y_2 = -L^{-1}\alpha(d/dt)y_1 - L^{-1}\beta y_1 - L^{-1}\gamma A_1$$

$$y_3 = -L^{-1}\alpha(d/dt)y_2 - L^{-1}\beta y_2 - L^{-1}\gamma A_2$$

$$\vdots$$

Solution of Van der Pol equation: For the Van der Pol equation, we have

$$y_0 = y(0) + ty'(0) + L^{-1}x(t)$$

$$y_1 = -L^{-1}\alpha(d/dt)y_0 - L^{-1}\beta y_0 - L^{-1}\gamma(d/dt)A_0$$

$$y_2 = -L^{-1}\alpha(d/dt)y_1 - L^{-1}\beta y_1 - L^{-1}\gamma(d/dt)A_1$$

etc.

16) Stochastic Oscillations: We could also have stochastic fluctuations in α, β, or γ in addition, of course, to stochastic $x(t)$ or initial conditions. Thus, in general we could write

$$\alpha = \langle\alpha\rangle + \varepsilon$$

$$\beta = \langle\beta\rangle + \eta$$

$$\gamma = \langle\gamma\rangle + \sigma$$

where ε, η, σ are zero-mean random processes. The solution process can now be obtained from

$$Ly = x - \alpha(d/dt)y - \beta y - \gamma \sum_{n=0}^{\infty} A_n$$

$$- \varepsilon(d/dt)y - \eta y - \sigma \sum_{n=0}^{\infty} A_n$$

where the A_n summation represents y^3 in the Duffing case and $(d/dt)y^3$ in the Van der Pol case. Thus,

$$y_0 = y(0) + ty'(0) + L^{-1}x$$

$$y_1 = -\alpha(d/dt)y_0 - \beta y_0 - \gamma A_0$$

$$\quad - \varepsilon(d/dt)y_0 - \eta y_0 - \sigma A_0$$

$$y_2 = -\alpha(d/dt)y_1 - \beta y_1 - \gamma A_1$$

$$\quad - \varepsilon(d/dt)y_1 - \eta y_1 - \sigma A_1$$

$$\vdots$$

Thus the summation of the above terms to some y_n is the solution of both oscillator equations except the A_n are different, of course, as explained above. Then $y(t) = \sum_{n=0}^{\infty} y_n(t)$ yields a stochastic series from which statistics can now be obtained easily by averaging without problems of statistical separability of quantities such as $\langle Ry \rangle$ where $R = \varepsilon(d/dt) - \eta$ which normally require closure approximations and truncations.

Remark: When transient behavior becomes oscillatory and periodic after a certain time, we can then consider it as a boundary-value problem. Sometimes one can obtain a solution for the oscillating behavior directly starting with the transient behavior. In other cases it may have to be approached separately.

These have been recently discussed by Noor and Atluri. In proposed large flexible space structures, supercomputer simulations will be important but are likely to be so large and complex that any means of reducing the computational effort needs to be a paramount objective. The discretization inherent in the computer techniques makes it difficult to see dependences or gain insight into the nature of the response and, further, enormously increases computation as grid spacing is decreased for accuracy.

The decomposition method provides continuous rather than discretized solution - a solution in analytic form - with an enormous potential saving in computation. This feature, as well as the natural, and physical, treatment of nonlinearity and stochasticity in the decomposition method, and its computability provide a powerful new tool for structural analyses.

17) Stochastic Structural Dynamics: The equation $Fu = g(t)$ can represent a generic problem in vibration where F is a nonlinear stochastic operator and g is a stochastic process. Suppose $Fu = Lu + Nu$ where L is a linear stochastic operator and Nu is a nonlinear (and possibly stochastic) term. L can be further decomposed into $L + R + R$ where L is a deterministic operator-specifically the highest-order linear differential operator - typically d^2/dt^2 in a vibration problem, R is the "remainder operator" representing the remaining portion of the linear and deterministic operator. (This device removes the necessity of

determination of the Green's function for the entire linear deterministic operator), and R indicates a stochastic operator.

If we consider, for example, the equation

$$m\ddot{y}(t) + a\,\dot{y}(t) + b\,y(t) = f(t)$$

and divide through by m, we can write now

$$\ddot{y}(t) + \alpha\,\dot{y}(t) + \beta\,y(t) = g(t)$$

Let $L = d^2/dt^2$, $R = \alpha\, d/dt$. Now we will assume for an example that α is constant but β is a random process $\beta(t)$ which quite reasonably, is assumed statistically independent of the input process $g(t)$. In the decomposition notation consistent with the notation in the literature, we let $R = \beta(t)$. We have

$$Ly + Ry + Ry = g(t)$$

$$Ly = g - Ry - Ry$$

$$y_0 = A + Bt + L^{-1}g$$

$$y_{n+1} = -L^{-1}Ry_n - L^{-1}Ry_n$$

y_0 is known as soon as specific initial-boundary conditions are given. All following components are determined for $n > 0$. Now $y = \sum_{n=0}^{\infty} y_n$ and an n-term approximation is given by $\phi_n = \sum_{i=0}^{n-1} y_i$. Now the

expectation of y is approximated by $\langle\phi_n\rangle$. Second-order statistics are also determinable - the correlation $K(t_1,t_2)$ is given by $\langle\phi_n(t_1)\phi_n(t_2)\rangle$. This method avoids all the usual restrictive assumptions and there is no statistical separability problem which ordinarily leads to closure approximations in averaging or hierarchy methods. Conventionally, stochastic structural dynamics can only allow the system input g to be stochastic. The decomposition method treats linear and nonlinear equations without closure approximations or perturbation. In addition, randomness can be considered in parameters or conditions. These cases are simply described in [2].

Orada and Haftka point out (in the AIAA Journal, vol. 25, no. 8, Aug. 1987, pp. 1133-1138) that in active vibration control, because of the interaction between the structure and the control system, simultaneous optimal design of both systems may be necessary. (More traditionally, the structure is optimized to minimize weight subject to the stress and stiffness constraints while the control system is optimized to minimize a performance measure involving deformation and control effort.) Such simultaneous optimization is discussed in the referenced paper using white noise inputs and linear control theory. We would point out only that a more powerful tool has now become available for such optimizations. We have seen now that the control theory can be made nonlinear and stochastic through application of the decomposition method without the use of numerical methods and we will be able to pursue this when funding is available.

The decomposition method allows more realistic models, eliminating restrictive mathematical assumptions. As an example, consider the

equation: $\ddot{x} + \alpha\dot{x} + \beta x = g$ where vibrating motion in the x direction, $\alpha\dot{x}$ is the damping force and βx is the restoring force. Initially the system is at rest so $x(0) = \dot{x}(0) = 0$. We can assume α, β, g, or any combination, to be stochastic. The widely-used averaging or hierarchy methods of determining statistics (seeking moments of the solution process directly) lead to unjustified closure approximations. Using decomposition, we do not need to require stationarity which is physically unlikely and mathematically restrictive. Perturbation methods work best in cases where fluctuations are small. In the perturbative case, one is assuming corrections to a deterministic solution are minor.

In a space borne effort costing many billions of dollars, it is essential that we seek physical solutions rather than mathematical solutions to models assuming processes that either do not exist in nature or exist in cases of minor vibrations. The mathematics must take into account any possibility of vibrations or effects which could lead to destabilization.

To recapitulate, we have here a second-order linear oscillator with stochastic parametric and external excitation. We can write

$$\ddot{x} + (\alpha_0 + \alpha')\dot{x} + (\beta_0 + \beta')x = g$$

where $\alpha_0 = \langle\alpha\rangle$, $\beta_0 = \langle\beta\rangle$, α', β', and g are statistically independent stochastic processes with given statistics to second-order. Conventional analyses assume white-noise processes which is neither physical nor necessary. Using L for d^2/dt^2, we have

$$Lx + \alpha_0\dot{x} + \beta_0 x + \alpha'\dot{x} + \beta' x = g$$

Let $R = \alpha_0\, d/dt + \beta_0$ and $R = \alpha'\, d/dt + \beta'$ to write

$$Lx + Rx + \mathit{R}x = g$$

or

$$Lx = g - Rx - \mathit{R}x$$

which is immediately solvable by decomposition as we have seen from previous examples. All that is necessary is operation on both sides with L^{-1}, decomposition of x into Σx_n, and identification of the x_0 term to include $L^{-1}g$ and the initial condition terms. It remains solvable even if we add a nonlinear term Nx (without assuming weak nonlinearity).

ACTIVE DAMPING OF RESPONSE OF LARGE SPACE STRUCTURES

Large space structures are necessarily flexible so that measures must be taken to minimize structural dynamic response to orbital corrections. The problem of active damping to transient response is considered, e.g., by Chen (*Int. J. Spacecraft, 21,* no. 5, Sept-Oct 1984, pp. 463-467). Let us consider a relevant model equation derived by Chen:

$$d^2u/dt^2 + \varepsilon|u|(du/dt) + u = 0$$

for which Chen finds an "equivalent linear" equation. By decomposition we instead write

$$Lu = -u - \varepsilon|u|(du/dt)$$

$$u = A + Bt - L^{-1}u - \varepsilon L^{-1}|u|(d/dt)u$$

$$u_0 = A + Bt$$

$$u_1 = -L^{-1}u_0 - \varepsilon L^{-1}A_0$$

$$u_2 = -L^{-1}u_1 - \varepsilon L^{-1}A_1$$

$$\vdots$$

where the A_n are derived for $|u|(d/dt)u$. Thus no linearization is necessary and we can see, in distinction to numerical solutions, how the solution depends on parameters and how a change in the model can affect the solution. In addition, of course, we can see the effect of linearization by comparing our solution with linearized solutions. We have seen in an earlier table (p. 31, 35) that the effects can be quite significant. We

are not commenting on this particular stiffness control scheme but only pointing out that we now have a better solution procedure for any such model. This, of course, was the objective of Phase I. In further work now, we can implement the new methodology on computers for application to real structures being designed by aerospace contractors.*

*Also we have seen how we can generalize control theory to apply to nonlinear, stochastic, and distributed cases by analytic rather than numerical procedures. This was not a part of our objective but it turned out an unexpected benefit of considerable significance.

A final example and general remarks:

As a basis of discussion and pertinent remarks, let us look at a rather general nonlinear partial differential equation of the type that could arise in some space application. Let us consider the (parabolic) equation*

$$u_t - \alpha u_{xx} + f(u,t,x)u_x = 0$$

on a defined finite region on R with $t \geq 0$. Assume α is a constant, f is a smooth function of t,x,u, and the initial/boundary conditions are given.

Of course, to obtain a quantitative solution, the specific form of f is required which is known when we have a particular model to solve. Since each model will be different, we have tried to show how any model can be calculated. When f is given, any separable terms in x and t will be designated by $-g(x,t)$ and any remaining term dependent on u, and multiplying u_x can be written $N(u,u_x)$. Let $L_t = \partial/\partial t$ and $L_{xx} = \partial^2/\partial x^2$ and write

$$L_t u - \alpha L_{xx} u = g - N(u,u_x)$$

The decomposition method solves for each linear operator term in turn; thus

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*In zero-gravity environments, heat transfer is thought to occur only by radiation or conduction. However, it is possible that heat transfer by convection can also occur, e.g., in heat pipe technology for cooling space stations sometimes used in satellites. This can occur if fluids are involved, perhaps in machinery. Rapid heating near a boundary can cause pressure waves producing a velocity and heat transfer by convection. For such a convection-diffusion process, we get a parabolic equation of this form.

$$L_t u = g + \alpha L_{xx} u - N(u,u_x)$$

$$L_{xx} u = -\alpha^{-1} g + \alpha^{-1} L_t u + \alpha^{-1} N(u,u_x)$$

The inverse L_t^{-1} for the operator L_t is an integration; similarly L_x^{-1} is a two-fold integration since L_x represents a two-fold differentiation. Since $L_t^{-1} L_t u = u - A = u - u(x,0)$ and $L_{xx}^{-1} L_{xx} u = u - B - Cx$, we obtain

$$u = A + L_t^{-1} g + \alpha L_t^{-1} L_{xx} u - L_t^{-1} N(u,u_x)$$

$$u = B + Cx - \alpha^{-1} L_{xx}^{-1} g + \alpha^{-1} L_{xx}^{-1} L_t u + \alpha^{-1} L_{xx}^{-1} N(u,u_x)$$

A is the initial condition, B and C are evaluated from the remaining two conditions. We add the two equations for u and divide by two, obtaining a single equation for u. If we define

$$u_0 = (1/2)\{u(x,0) + B + Cx + L_t^{-1} g - \alpha^{-1} L_{xx}^{-1} g\}$$

$$K = -(1/2)\{\alpha L_t^{-1} L_x + \alpha^{-1} L_{xx}^{-1} L_t\}$$

$$G = -(1/2)\{L_t^{-1} - \alpha^{-1} L_{xx}^{-1}\}$$

we have

$$u = u_0 + Ku + G \cdot N(u,u_x)$$

The solution by decomposition is $u = \sum_{n=0}^{\infty} u_n$ with the given u_0 and the remaining components given by

$$u_{n+1} = Ku_n + GA_n$$

for $n \geq 0$, where the A_n are the polynomials defined by the author

in [1] for $N(u,u_x)$, i.e., $N(u,u_x) = \sum_{n=0}^{\infty} A_n$. For analytic nonlinearities for which the A_n can be determined and for specified initial/boundary conditions leading to a convergent series, the solution is $u = \sum_{n=0}^{\infty} u_n$, although an n-term approximation $\phi_n = \sum_{i=0}^{n-1} u_i$ for some n serves as a practical solution, usually with relatively small n, as seen in [1]. The solution can also be made by operating on the equation $L_t u - \alpha L_{xx} u = g - N(u,u_x)$ with $(L_t^{-1} - \alpha^{-1} L_{xx}^{-1})$.

Though the result is the same and ϕ_t, ϕ_x are evaluated from the given conditions, writing u_0 as

$$u_0 = (1/2)(\phi_t + \phi_x) + (1/2)(L_t^{-1} - \alpha^{-1} L_{xx}^{-1})g$$

means the result is not limited to the given parabolic equation - the derivatives could be of any order - the operator L represents the highest-order derivative. In this particular case we used the double subscript L_{xx} to denote a second-order derivative. The A_n can be generated for wide classes of composite nonlinear functions [1,2]. Thus the method is quite global.

When solutions are obtained, we find they converge, generally with extreme rapidity, so a few terms of $u = \sum_{n=0}^{\infty} u_n$ suffice, i.e., we use a practical (n-term) approximation $u \simeq \sum_{i=0}^{n-1} u_i$. If we take n very large, this approximation approaches $\sum_{i=0}^{\infty} u_i$ which is the definition of u. We are dealing with physical systems with bounded inputs, analytic nonlinearities for which the A_n can be determined, and specified initial/boundary conditions, assuming integrability of the forcing

function - all physically reasonable conditions to get convergence.

In stochastic cases, the series is stochastic and from the approximate u, we can determine expectation or covariance.

INTRODUCTION TO THE PROBLEM OF CONTROL

The control problem involves not just solving a differential equation - the state space equation - but finding a solution satisfying a performance functional or criterion to be minimized (such as least fuel, least cost, least time, etc.). Bellman has discussed the differential equation $dy/dt = g(y,r)$, $y(0) = c$ and the criterion equation

$$J(u,v) = \int_0^T h(y(t_1), r(t_1))dt_1 + k(y(T))$$

viewing $r(t)$ as a point in function space and setting the variation of J equal to zero to obtain a necessary condition for the determination of $r(t)$. This procedure yields the Euler equation and some auxiliary conditions which convert the problem of minimizing the integral functional to that of solving a set of differential equations for y and r. These equations are nonlinear and subject to two-point boundary conditions.

Since the decomposition method has been demonstrated to yield solutions for formerly analytically intractible nonlinear or stochastic equations, it is clear that a viable approach to the general control problem is possible. The difficulty previously was not the formulation of the Hamiltonian approach but the inability to solve stochastic or nonlinear equations as they are or a global approach to partial differential equations. The problem has now been shown to fit in the format of the decomposition method.

Thus one can consider a state space equation of the form $\dot{x} = f(x_1,\ldots,x_n; u_1,\ldots,u_m; t)$ and general performance criterion functions J extending to the stochastic and multidimensional cases for application to space structures. The next section considers our approach to the analysis.

AN OUTLINE OF OUR APPROACH TO CONTROL

Suppose we consider a nonlinear, possibly stochastic which we want to control in some optimal way. For a linear control system with a quadratic performance index, of course an analytical solution can be made. Consider the state equations

$$\dot{x}(t) = f(x_1,\ldots,x_n;\ u_1,\ldots,u_m;\ t)$$

i.e., a set of n nonlinear differential equations with $x(t)$ representing a state vector with n components $f_1,\ldots,f_n$, and $x(t_0)$ a given initial vector. Define, for example [4] a performance functional $J(x,u,t)$ given by

$$J = \phi[x(t_1),\ t_1] + \int_{t_0}^{t_1} F(x,u,t)\ dt$$

where ϕ and F are scalar functions with necessary smoothness properties. Let $p = [p_1,\ldots,p_n]^T$ be a vector of Lagrange multipliers and form an augmented functional

$$J' = \phi[x(t_1),\ t_1] + \int_{t_0}^{t_1} [F(x,u,t) + p^T\ (f-\dot{x})]\ dt$$

Integration by parts leads to

$$J' = \phi - [p^T\ x]\ \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} [H + \dot{p}^T\ x]\ dt$$

with H defined as

$$H(x,u,t) = F(x,u,t) + p^T f$$

If u is defined on $t_0 \le t \le t_1$, we vary u and find the variation $\delta J'$ corresponding to δu, leading to the n adjoint equations,

$$\dot{p}_i = - \partial H/\partial x_i$$

so we have a system of $2n$ nonlinear differential equations with two-point boundary conditions. Although this approach has been discussed by R. E. Bellman and many others perhaps most recently in [4], analytical solution has usually not been possible except by numerical methods. We now have a promising and potentially valuable alternative since such systems of *nonlinear* differential equations have been solved (even for the stochastic and/or multidimensional cases) in an analytic approximation by the decomposition method [1-3].

Another possibility is through solution by decomposition of the matrix Riccati equation which appears in invariant embedding and neutron transport theory as well as modern control theory. Consider

$$R'(x) = B(x) + D(x)R(x) + R(x)D(x) + R(x)B(x)R(x)$$
$$R(0) = 0$$

where B, D, R are continuous $n \times n$ non-negative matrices. Suppressing the argument x, we have

$$R' = B + DR + RD + RBR$$

If $L = d/dx$

$$LR = B + HR + NR$$

where $LR = R'$, $HR = DR + RD$, and NR represents a nonlinear operator on R. Since $R(0) = 0$, operation with L^{-1} on both sides yields

$$R = L^{-1}B + L^{-1}HR + L^{-1}NR.$$

Let R and NR be written in terms of the A_n polynomials. For R this is equivalent to writing $R = \sum_{n=0}^{\infty} R_n$. For NR we write $\sum_{n=0}^{\infty} A_n$. Identify $R_0 = L^{-1}B$; then

$$R_0 = L^{-1}B$$

$$R_1 = L^{-1}HR_0 + L^{-1}A_0$$

$$R_2 = L^{-1}HR_1 + L^{-1}A_1$$

$$\vdots$$

$$R_n = L^{-1}HR_{n-1} + L^{-1}A_{n-1}$$

for $n \geq 1$. The A_n for NR are given by [1]

$$A_0 = R_0BR_0$$

$$A_1 = R_0BR_1 + R_1BR_0$$

$$A_2 = R_1BR_1 + R_0BR_2 + R_2BR_0$$

$$A_3 = R_0BR_3 + R_3BR_0 + R_1BR_2 + R_2BR_1$$

$$A_4 = R_2BR_2 + R_0BR_4 + R_4BR_0 + R_1BR_3 + R_3BR_1$$

$$\vdots$$

so that

$$R_0 = L^{-1}B$$

$$R_1 = L^{-1}HR_0 + L^{-1}R_0BR_0$$

$$R_2 = L^{-1}HR_1 + L^{-1}(R_0BR_1 + R_1BR_0)$$

$$R_3 = L^{-1}HR_2 + L^{-1}(R_1BR_1 + R_0BR_2 + R_2BR_0)$$

$$\vdots$$

Finally, since $HR = DR + RD$

$$R_0 = L^{-1}B$$

$$R_1 = L^{-1}(DR_0 + R_0D) + L^{-1}(R_0BR_0)$$

$$R_2 = L^{-1}(DR_1 + R_1D) + L^{-1}(R_0BR_1 + R_1BR_0)$$

$$R_3 + L^{-1}(DR_2 + R_2D) + L^{-1}(R_1BR_1 + R_0BR_2 + R_2BR_0)$$

$$\vdots$$

An n-term approximant is $\phi_n = \sum_{i=0}^{n-1} R_n$ which approaches $R = \sum_{i=0}^{\infty} R_n$ as $n \to \infty$. Thus given B, D, a specific R can be calculated to a desired approximation. Accuracy has been demonstrated in [5].

FINAL REMARKS

When we study a physical system, we must simplify the actual system with an idealized model represented by a set of equations which can then be solved by available techniques. Thus, modeling is always a compromise between a sufficiently realistic representation and tractability of the resulting equations. The real test is whether the resulting solution yields results in close conformity with the actual physical system. As our ability to solve more complicated equations increases, we become less limited in making our models physically accurate. Thus, as computers have grown in capability and speed, there has also developed an increasing reliance on them. This dependence has in some cases given rise to an attitude that analytical solutions are neither possible nor really desirable. It is indeed correct that closed form solutions are generally not possible, and that everything else can be considered "approximation." Yet, *all* modeling is approximation, and closed form solutions mean very little. Linearization, for example, is a general procedure and has contributed much, but the linearized model is not the same problem and may lose important aspects of the real behavior. When we consider that closed form solutions result from *changing* the physical problem until the equations are sufficiently simple, we realize that we are solving a *different* problem.

We have demonstrated that the decomposition method allows physically realistic and accurate solutions of the types of vibration, heating, and control problems which will arise in the analysis of large space structures planned for the next two or three decades. The severe requirements imposed by the sizes, weights, necessary stiffness, and specifications on thermal

and mechanical distortions make this methodology essential because of its particular features.

The chief advantage is that solution of nonlinear problems and/or random problems is possible without linearization, assumption of weak nonlinearity, perturbation, assumption of special (and unphysical stochastic) processes, or classic numerical methods. The elimination of the restrictive assumptions and limitations of usual methods will reduce computational effort and add certainty to the design optimization process of rapidly maneuvering space structures!

It is essential in dealing with the complex interdependent problems arising in contemplated space structures that we make physically realistic solutions, not mathematical solutions of models which have been made solvable by restrictive assumptions. Numerical solutions by super-computers have brought us far but have drawbacks of sometimes excessive computer time and a läck of insight into functional dependences. We need insight into performance and correct solutions of the physical problems to prevent failure in operation. The decomposition method allows us a new and valuable alternative to be used by aerospace contractors.

A LOOK AHEAD

So far, we have shown in the analysis of problems arising in space structures that the decomposition method provides efficiency and accuracy as well as other important advantages. These advantages are insight into functional dependences, physically more correct and realistic solutions, and a considerable potential savings of computational effort. This work applies to the response of nonlinear structures involving random vibrations, fluctuations of parameters, thermal response in problems involving random or temperature-dependent thermal conductivity, radiative transfer, and control.

Two additional avenues have become apparent. First, a major generalization and extension of control theory to the analytical solution of the nonlinear stochastic multidimensional case has now become possible.

Second, programming of the decomposition method is essential now. We believe this will be a major step with significant advantages over numerical methods now in use.

Both of these are of crucial significance to programs such as those contemplated by SDIO. The computer implementation of decomposition will allow us to consider problems of greater complexity and general contractor models. The control theory will allow us to do the problems of distributed nonlinear stochastic control essential for space structures. Both should be pursued as separate projects to be brought together for Phase III type work.

The overall program is intended to allow the solution of any specific problems of interest in real space structure models of interest to Air Force, NASA, or space station contractors.

REFERENCES ON DECOMPOSITION METHOD

1) G. Adomian, Stochastic Systems, Academic Press, 1983.

2) G. Adomian, Nonlinear Stochastic Operator Equations, Academic Press, 1986.

3) R. Bellman and G. Adomian, Nonlinear Partial Differential Equations - New Methods for Their Treatment and Applications, Reidel, 1985.

4) G. Adomian, "Applications of the Decomposition Method to the Navier-Stokes Equations," *J. Math. Anal. and Applic., 119,* nos. 1/2, October/November 1986.

5) G. Adomian, "Decomposition Solution for Duffing and Van der Pol Oscillators," *Int. J. of Math. Sciences, 9,* no. 4, 1986: 731-2.

6) N. Bellomo and R. Riganti, "Semilinear Stochastic Systems: Analysis with the Method of the Stochastic Green's Function and Application in Mechanics," *J. Math. Anal. and Applic., 96,* no. 2, 30 October 1983.

7) N. Bellomo and R. Monaco, "A Comparison Between Decomposition Methods and Perturbation Techniques for Nonlinear Random Differential Equations," *J. Math. Anal. and Applic., 110,* no. 2, September 1985.

8) I. Bonzani, "On a Class of Nonlinear Stochastic Dynamical Systems: Analysis of the Transient Behavior," *J. Math. Anal. and Applic.,* in press.

9) R. Rach, "A Convenient Computational Form for the A_n Polynomials," *J. Math. Anal. and Applic., 102,* no. 2, September 1984: 415-419.

10) E. Gabetta, "On a Class of Semilinear Stochastic Systems in Mechanics with Quadratic Type Nonlinearities: Time Evolution of the Probability Density," *J. Math. Anal. and Applic.,* in press.

11) R. Rach, "On Continuous Approximate Solutions of Nonlinear Differential Equations," *J. Math. Anal. and Applic.,* in press.

12) R. Rach, "On the Adomian (Decomposition) Method and comparisons with Picard's Method." *J. Math. Anal. and Applic.,* in press.

13) D. Bigi and R. Riganti, "Solutions of Boundary-Value Problems by Decomposition Method,"

14) Bellomo, N., de Socio, L., and Monaco, R., "On the Random Heat Equation: Solution by the Stochastic Adaptive Interpolation Method," *Comp. and Math. with Appl.,* 1987, in press.

15) N. Bellomo, R. Cafaro, G. Rizzi, "On the Mathematical Modeling of Physical Systems by Ordinary Differential Stochastic Equations," *Math. & Comp. in Simulation, 4* (1984):361-367.

16) R. Riganti, "Transient Behavior of Semilinear Stochastic Systems with Random Parameters," *J. Math. Anal. and Appl., 98* (1984):314-327.

17) I. Bonzani, "Analysis of Stochastic Van der Pol Oscillators Using the Decomposition Method," in Complex and Distributed Systems - Analysis, Simulation and Control (S. Tzafestas & P. Borne, Eds.), North Holland, Amsterdam (1986):163-168.

18) N. Bellomo, R. Riganti, Nonlinear Stochastic Systems in Physics and Mechanics, World Scientific Publ., Singapore (1987).

19) R. Riganti, "Analytical Study of a Class of Nonlinear Stochastic Autonomous Oscillators With One Degree of Freedom," *Meccanica, 14,* no. 4, 1979:180-86.

20) N. Bellomo and D. Sarafyan, "On Adomian's Decomposition Method and Some Comparisons with Picard's Iterative Scheme, *J. Math. Anal. and Applic., 123,* no. 2, May 1, 1987:389-400.

21) R. Riganti, "On a Class of Nonlinear Dynamical Systems: The Structure of a Differential Operator in the Application of the Decomposition Method," *J. Math. Anal. and Applic., 124,* no. 1, May 15, 1987:189-199.

22) I. Bonzani and R. Rigante, "Periodic Solutions of Nonlinear Dynamical Systems by the Decomposition Method," *Mech. Res. Comm.*, Pergamon 1987, in press.

23) G. Adomian, Applications of Nonlinear Stochastic Systems Theory to Physics, Reidel, 1988.

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